Index theory and bulk-boundary correspondence of topological insulators and semimetals

Indextheorie und Rumpf-Rand-Korrespondenz von topologischen Isolatoren und Halbmetallen

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Introduction

Since the discovery of the Integer Quantum Hall effect and its subsequent explanation in terms of the Chern number of a vector bundle [129] there has been a tremendous mathematical interest in topological phases of matter. In such phases topological quantum numbers give rise to unconventional macroscopic behavior such as quantized transport coefficients or protected states at boundaries and interfaces. Some of these boundary states are of particular interest since they may host exotic quantum states that cannot occur in the bulk of a material. The existence these boundary states and their stability under disorder can be enforced by non-vanishing topological invariants in the bulk via a general principle of bulk-boundary correspondence, which has been established rigorously for large classes of lightly disordered and aperiodic topological insulators [103][79][53][29][27]. For these systems the existence of a gap in the spectrum allows the use of powerful mathematical tools from K-theory and non-commutative geometry. The situation is very different for materials in which the disorder is so strong that the spectral gap is closed in the bulk and replaced by a so-called mobility gap. There are only few rigorous results available for the bulk-boundary correspondence of strongly disordered topologically insulators [51][60][119][109][111]. In the recent work [111] we found that besides the strongly disordered insulators there is another class of gapless materials which can carry non-trivial bulk invariants: Topological semimetals, which have in their spectrum some energy value at which the density of states vanishes exactly; we then say that there is a pseudogap at that energy. The best-known examples are Dirac- and Weyl-semimetals in which the valence and conductance bands touch linearly in isolated points, i.e. the low energy excitations can be described by the Dirac/Weyl-Hamiltonian [11]

$$H = \vec{\sigma} \cdot \vec{p},$$

but there are also other possibilities such as nodal-line-semimetals, where the bands touch along a line in momentum space. While such band touching points occur generically in some systems, they are much less stable than spectral gaps. Nevertheless, topological semimetals do exhibit interesting surface states such as the flatband states of chiral graphene or the Fermi-arc states of Weyl-semimetals. However, the well-definedness of disordered topological semimetals as a phase of matter and the robustness of their surface states is questionable (see [131]).

This thesis extends an operator-algebraic framework developed for topological quantum systems [100][102][103] to derive index theorems for mobility-gapped and pseudogapped models and results on bulk-boundary correspondence for low-dimensional topological invariants. This thesis can be seen as the continuation of a program started with the recent monograph [111] which derived part of the present results for tight-binding models. In contrast with [111] we consider here a more general setup that also encompasses continuous models of topological phases. This leads to new complications which come from the consideration of unbounded Hamiltonians and the fact that it sometimes does not make sense to associate bulk topological invariants to a single system, but that one must instead choose a reference system for comparison. Accordingly, there are also some new results for the bulk-boundary correspondence of models with a spectral gap in the bulk models and a more general definition for bulk topological invariants to accommodate the more varied phenomenology allowed by continuous models compared to tight-binding models.

We now summarize the individual chapters:

- Chapter 1 covers some preliminary material, much of which is well-known but there are also some substantial new technical results. A major theme of this work are C^*- or W^*-algebras which carry an abelian n-parameter group action and an invariant trace. Therefore we recall basic facts on C^*- and W^*-dynamical systems as well as their associated crossed product algebras. We then turn to a more analytic side and discuss the interplay of group actions with non-commutative L^p-spaces, in particular different notions of differentiability and non-commutative analogues of Sobolev and Besov spaces following [111]. This part is followed by a discussion of differentiable operators on Hilbert modules, which is not only review but contains some essentially new material. In particular we study a new notion of smoothness for unbounded self-adjoint operators on a Hilbert module which is strong enough to prove that the bounded transform maps unbounded smooth operators into strictly smooth operators. Finally we recall some basic notions of K-theory, explain how K-theory classes can be represented by pairs of elements in the stable multiplier algebra and describe the numerical pairings of K-theory with cyclic cohomology.
- Chapter 2 treats semifinite index theorems for abelian n-parameter group actions as higher-dimensional non-commutative analogues of the Gohberg-Krein theorem based on [111]. The goal is to write a geometric invariant, namely the non-commutative Chern number of a projection or unitary in some algebra as the index of a related operator in a semifinite von Neumann algebra. The chapter begins with a more detailed treatment of crossed

products with respect to n-parameter group actions and explains how they can be used to perform analysis on covariant representations, i.e. when the group action is given by conjugation with a spatially represented group of unitaries. Then we briefly recall the criteria for quantum differentiability in terms of Sobolev- and Besov spaces derived in [111] and explain how those imply the index theorems for symbols which have a certain kind of Sobolev/Besov-regularity. This is followed by some essentially new material which shows how to write down semifinite index theorems for the pairings with K-theory elements in the multiplier picture.

- Chapter 3 reintroduces the smooth Toeplitz extension, which is an exact sequence associated to any C^*-algebra with a strongly continuous one-parameter group action. We recall material from [111] where it was identified as an abstract version of the bulk-boundary exact sequences that are used in solid state physics and its connecting maps in K-theory described concretely in terms of the Connes-Thom isomorphisms. In particular, for a C^*-algebra with an additional n-parameter group action and an associated trace one has natural pairings with Chern cocycles which are dual w.r.t. the connecting maps. The final section introduces a new exact sequence to be used for bulk-interface correspondence in the applications and whose connecting maps are described in terms of the smooth Toeplitz extension. We then look at the related issue of how to efficiently compute the pairing of a Chern cocycle with a K-theory class given in the multiplier picture and derive a simple formula for an important special case.

- Chapter 4 begins the main part of this thesis. It introduces an abstract operator-algebraic setting for solid state physics based on observable algebras which are tracial dynamical systems consisting of C^*-algebras $\mathcal{A}$ with a $\mathbb{R}^d$-action and an invariant trace. In a physical representation, the action is generated by exponentiation of d commuting position operators and the trace is an averaged trace per unit volume. The goal of the chapter is to assign topological invariants to Hamiltonians, i.e. self-adjoint multipliers of an observable algebra, either as elements of a K-group $K_i(\mathcal{A})$ or purely numerical as an index pairing of the type described in Chapter 2. Those topological invariants fall into two categories: gapped invariants which we associate to Hamiltonians with a spectral gap or gapless invariants which obstruct the formation of such gap. The definition of gapped invariants for general multipliers is not always possible, or at least not sensible, for the simple reason that spectral projections of multipliers do not always define classes in $K_0(\mathcal{A})$ on their own. We handle this with a formalism of reference Hamiltonians where the gapped invariants are defined only for pairs of Hamiltonians with a common spectral gap as the formal difference of their Fermi projections. Conditions are also discussed under which the dependence on a reference Hamiltonian can be eliminated. From [111] we further know that the gapped invariants in tight-bindings models are still well-defined and admit semifinite index theorem if there is no spectral gap but only a mobility gap or pseudogap. This result is generalized to the present setting for Hamiltonians that are perturbations of a spectrally gapped reference Hamiltonian. Finally we discuss gapless invariants and their basic properties. It is recalled how to associate a class in $K_i(\mathcal{A})$ to an $\mathcal{A}$-multiplier which is invertible modulo the ideal $\mathcal{A}$ and defines Chern numbers of that class as numerical invariants. It is shown that non-triviality of any of those Chern numbers not only obstructs the existence of a spectral gap, but also of a mobility or pseudogap. This generalizes recent results that the non-triviality of edge Chern numbers prohibits dynamical localization at the surface of a topological insulators [103, 29, 111].

- Chapter 5 discusses the algebraic approach to bulk-boundary and bulk-interface correspondence of topological insulators based on the connecting maps in K-theory. The stated goal is to relate the gapless topological invariants of a Hamiltonian on a system with boundary to the gapped invariants of a bulk Hamiltonian with a spectral gap. It is clarified that the bulk-boundary correspondence must be modified if the bulk Hamiltonian is only a multiplier of the bulk algebra, in particular there can in general only be a relative bulk-boundary correspondence which compares the boundary invariants of one Hamiltonian to those of a reference Hamiltonian. But even such a relative bulk-boundary correspondence does not always hold. As recent examples from continuum models show [125][59], the boundary modes can depend heavily on the choice of boundary condition, i.e. self-adjoint extension of the boundary Hamiltonian, and not all of them fit together with K-theoretic approach. A distinguished family of self-adjoint extensions of halfspace Hamiltonians is introduced, the so-called resolvent-affiliated extensions, for which a relative bulk-boundary correspondence can be formulated and is stable. After some examples we discuss bulk-interface correspondence based on an exact sequence introduced in Chapter 3.

- Chapter 6 finally discusses bulk-boundary correspondence in special cases where the bulk Hamiltonian does not have a spectral gap but only a mobility or pseudogap. The first part on flat bands of surface states shows a relation between the existence of zero-modes of a chiral Hamiltonian on a halfspace with a non-commutative winding number of its corresponding bulk Hamiltonian. It generalizes one of the main results of [111] (which in turn was an improvement of the results of my master thesis [118]) from tight-binding models to certain unbounded Hamiltonians. The second part proves a regularized bulk-interface correspondence which relates differences of bulk two-dimensional Chern numbers across an interface to currents perpendicular to said interface. The main line of the argument follows [51] which performed that analysis for two-dimensional tight-binding Hamiltonians. We will use very similar regularizations adapted for compatibility with our algebraic setting and to allow unbounded Hamiltonians as well as a pseudogaps instead of a mobility gap. The final part of the chapter discusses examples, in particular a class of continuum models is exhibited which satisfies some of the more difficult to check conditions of the main results.

Zusammenfassung

In dieser Dissertation wird der Operator-algebraische Ansatz zur Beschreibung von topologischen Quantensystemen aus [100][102][103] weiterentwickelt, in diesem Rahmen werden Indextheoreme für Modelle mit Mobilitätslücken und Pseudobandlücken diskutiert ebenso wie Ansätze für die Rumpf-Rand-Korrespondenz von niedrig-dimensionalen topologischen Invarianten. Die vorliegende Arbeit kann man verstehen als Fortsetzung von einem Programm, das in der Monographie [111] begonnen wurde und welche viele der hiesigen Resultate im Rahmen spezieller Tight-Binding-Modelle beweist. Im Gegensatz zu [111] beschäftigen wir uns hier nicht nur mit Tight-Binding-Modellen, sondern sind allgemeiner und diskutieren viele der Besonderheiten von kontinuierlichen Modellen topologischer Phasen. Wesentliche Komplikationen dabei sind, dass solche Kontinuumsmodelle durch unbeschränkte Hamilton-Operatoren beschrieben werden und dass es manchmal nicht sinnvoll ist, einzelnen Systemen topologische Invarianten zuzuweisen, im Allgemeinen muss man wenigstens ein weiteres System als Bezugspunkt wählen. Diese Komplikationen wirken sich auch schon auf die Rump-Rand-Korrespondenz für topologische Isolatoren mit spektraler Bandlücke, für welche wir daher ebenfalls wesentlich neue Resultate beweisen.

Es folgt ein Überblick über die einzelnen Kapitel:

- Kapitel 1 behandelt allgemeinere Themen, von denen die meisten wohlbekannt sind, aber es gibt auch ein paar wesentlich neue technische Resultate. Eine übergeordnete Thematik dieses Werkes sind C^*- und W^*-Algebren auf denen abelsche n-Parametergruppe wirken und die mit einer invarianten Spur ausgestattet sind. Daher wiederholen wir einige Grundlagen zu C^*- und W^*-dynamischer Systemen sowie deren gekreuzten Produkten. Dann wenden wir uns eher analytischeren Themen zu, wie dem Zusammenspiel von Gruppenwirkungen und nichtkommutativen L^p-Räumen. Es werden unterschiedliche Differenzierbarkeitsbegriffe diskutiert und darauf aufbauend Sobolev- und Besov-Räume. Darauf folgt eine Betrachtung differenzierbarer Operatoren auf einem Hilbertmodul, die einen neuen Glattheitsbegriff für unbeschränkte Operatoren einführt, der stark genug ist um die Glattheit der beschränkten Transformation ebensolcher sicherzustellen. Schließlich erinnern wir an Grundlagen der K-Theorie, erklären wie K-Theorie-Klassen durch Paare von Elementen in der stabilen Multiplikatoralgebra dargestellt werden können und beschreiben die Paarung von K-Theorie mit zyklischer Kohomologie.

- Kapitel 2 behandelt semifinite Indextheoreme für Wirkungen abelscher n-Parametergruppen als höherdimensionale nichtkommutative Analoga des Satzes von Gohberg-Krein und basiert größtenteils auf [111]. Das Ziel ist es eine geometrische Invariante, die nichtkommutative Chernzahl, einer Projektion oder einer Unitären in einer Algebra als den Index eines entsprechenden Operators in einer semifiniten von-Neumann-Algebra zu schreiben. Das Kapitel beginnt mit einer detaillierteren Beschreibung der gekreuzten Produkte bezüglich n-Parametergruppen und erklärt wie man diese verwenden kann zur Analyse von kovarianten Darstellungen. Dann wiederholen wir kurz die Sobolev- und Besov-Kriterien für Quantendifferenzierbarkeit, die in [111] bewiesen wurden, und erklären wie diese Indextheoreme für Symbolklassen mit ausreichender Regularität implizieren. Den Abschluss bildet ein Abschnitt mit neuem Material welches zeigt, wie man semfinite Indexpaarungen für K-Theorie Klassen schreiben kann, wenn für diese nur Repräsentanten im Multiplikatorbild vorliegen.

- Kapitel 3 beginnt mit einer erneuten Betrachtung der glatten Toeplitzerweiterungen, die man aus jeder C^*-Algebra mit einer stark stetigen Einparamatergruppenwirkung konstruieren kann. Es wird erinnert an Material aus [111], wo diese als abstrakte Version der Rumpf-Rand exakten Sequenzen aus den Anwendungen in der Festkörperphysik identifiziert wurden, insbesondere an die Verbindungsabbildungen in K-Theorie, welche eng mit den Connes-Thom-Isomorphismen zusammenhängen und dual sind bezüglich der Paarungen mit Chernkozykeln. Der letzte Abschnitt motiviert eine neue exakte Sequenz für Rumpf-Grenzflächen-Korrespondenz welche in den späteren Anwendungen benutzt wird. Es wird auch darauf eingegangen, wie man die Paarung eines Chernkozykel mit einer K-Theorie Klasse im Multiplikatorbild effizient berechnen kann.

- Kapitel 4 beginnt den Hauptteil dieser Arbeit. Es führt einen abstrakten Operator-algebraischen Rahmen ein basierend auf Observablenalgebren $\mathcal{A}$, die mit einer $\mathbb{R}^d$-Wirkung und einer invarianten Spur ausgestattet sind. In physikalischen Darstellungen wird die Wirkung erzeugt durch d kommutierende Ortsoperatoren und die Spur entspricht einer gemittelten Spur pro Volumen. Das Ziel ist es, Hamiltonoperatoren, d.h. selbstadjungierten Multiplikatoren von $\mathcal{A}$, topologische Invarianten zuzuordnen, entweder als Elemente einer K-Gruppe $K_i(\mathcal{A})$ oder als rein numerische Invarianten als Indexpaarung wie in Kapitel 2. Diese topologischen Invarianten fallen in zwei Kategorien: Invarianten von Systemen mit Bandlücke und lückenlose

Invarianten, welche gerade die Formation einer solchen Bandlücke verhindern. Im Fall mit Bandlücke ist es nicht immer möglich, oder zumindest nicht sinnvoll, einem allgemeinen Multiplikator Invarianten zuzuordnen, einfach weil die Spektralprojektionen eines solchen nicht immer Klassen in $K_0(\mathcal{A})$ definieren. Dies wird umgangen durch einen Formalismus von Referenz-Hamiltonians, bei dem Invarianten nur definiert sind für Paare von Hamiltonians mit gemeinsamer Bandlücke als formale Differenz ihrer Fermi-Projektionen. Es werden auch Bedingungen diskutiert, in denen man auf einen solchen Referenz-Hamiltonian verzichten kann. Aus [111] wissen wir bereits, dass die Invarianten in Tight-Binding-Modellen immer noch wohldefiniert sind und semifinite Indextheoreme erlauben, wenn man keine spektrale Bandlücke hat, sondern nur eine Mobilitätslücke oder eine Pseudobandlücke. Dieses Resultat wird für die vorliegende Situation verallgemeinert für Hamiltonians, die geeignete Störungen eines Referenz-Hamiltonians mit echter Bandlücke sind. Dann werden lückenlose Invarianten diskutiert. Es wird wiederholt, wie man einem $\mathcal{A}$-Multiplikator welcher invertierbar ist modulo $\mathcal{A}$ eine Klasse in $K_i(\mathcal{A})$ zuweist und definieren wieder Chernzahlen als numerische Invarianten. Es wird dann bewiesen, dass nicht-triviale lückenlose Chernzahlen nicht nur Bildung einer spektralen Lücke, sondern ebenso einer Mobiltäts- oder Pseudobandlücke verhindern. Dies verallgemeinert neuere Resultate, wonach die Nicht-Trivialität von Rand-Chernzahlen dynamische Lokalisierung von Randzuständen verbietet.

- Kapitel 5 widmet sich dem algebraischen Ansatz zur Rumpf-Rand- und Rumpf-Grenzflächen-Korrespondenz von topologischen Isolatoren aufbauend auf den verbindenden Abbildungen der K-Theorie. Das erklärte Ziel ist es, die lückenlosen Invarianten eines Hamiltonian für ein System mit Rand in Verbindung zu setzten mit denen des asymptotischen Rumpfsystems mit Bandlücke. Es wird klargestellt, dass die Rumpf-Rand-Korrespondenz im Allgemeinen angepasst werden muss: Wenn der Rumpf-Hamiltonian nur ein Multiplikator der Rumpf-Algebra ist, kann man oft nur eine relative Korrespondenz formulieren, bei der je zwei Systeme mit Rand und im Rumpf verglichen werden. Aber auch solche relativen Resultate decken die ganze Phänomenologie nicht ab: Wie kürzliche Beispiele von Kontinuumsmodellen zeigen [125][59], hängen die auftretenden Randzustände stark von den Randbedingengen ab, d.h. von der Wahl der selbstadjungierten Erweiterung eines Halbraum-Hamiltonian, und nicht für alle Wahlen ist K-Theorie anwendbar. Es wird eine Familie von selbstadjungierten Erweiterungen charakterisiert, die so-genannten Resolventen-affiliierten Multiplikatoren, für welche eine relative Rumpf-Rand-Korrespondenz formuliert werden kann und stabil ist. Nach einigen Beispielen wird auch kurz eine Rumpf-Grenzflächen-Korrespondenz diskutiert, basierend auf der exakten Sequenz aus Kapitel 3.

- Kapitel 6 diskutiert schließlich Rumpf-Rand-Korrespondenz in Spezialfällen in denen der Rumpf-Hamiltonian keine spektrale Bandlücke besitzt, sondern nur Mobilitäts- oder Pseudolücke. Der erste Teil zeigt einen Zusammenhang zwischen flachen Bändern von Randzuständen eines chiralen Hamiltonian mit einer nichtkommutativen Windungszahl im Rumpf. Es verallgemeinert eines der Hauptresultate von [111] (welches wiederum auf der Masterarbeit des Autors [118] aufbaut) von Tight-Binding-Modellen zu bestimmten unbeschränkten Hamilton-Operatoren. Der zweite Teil beweist eine regularisierte Rumpf-Grenz-flächen-Korrespondenz, welche die Differenz von zwei-dimensionalen Rumpf-Chernzahlen auf zwei Seiten einer Grenzfläche in Verbindung setzt zu Strömen die senkrecht zu derselben fließen. Die Argumentation folgt dabei stark [51] welches diese Analyse erstmals durchführte für zwei-dimensionale Tight-Binding-Modelle mit Mobilitätslücke. Wir verwenden hier eine sehr ähnliche Regularisierung, welche an unseren algebraischen Rahmen angepasst wird und damit auch schwache Chernzahlen, unbeschränkte Hamiltonians und Pseudolücken abdeckt. Der letzte Abschnitt diskutiert Beispiele; insbesondere demonstrieren wir, dass es eine Klasse von Kontinuumsmodellen gibt, die die teils schwierig nachzuprüfenden Regularitätsbedingungen der Hauptresultate erfüllen.

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1 Preliminaries

1.1 Dynamical systems and crossed products

We recall some basic notions about C^*- and W^*-dynamical systems and their associated crossed product algebras based on the standard references [92, 122, 21]. In this work we only need abelian groups, therefore G will always be a locally compact abelian group with additively written operation.

Definition 1.1.1 *A C^*-dynamical system is a triple $(\mathcal{A}, G, \alpha)$ consisting of a C^*-algebra $\mathcal{A}$ and a strongly continuous G-action $\alpha : G \to \mathrm{Aut}(\mathcal{A})$, namely the orbit $t \in G \mapsto \alpha_t(a)$ is norm-continuous for each $a \in \mathcal{A}$.*

A covariant representation of a C^*-dynamical system is a pair (π, U) with π a non-degenerate representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_0$ and U a strongly continuous unitary (projective) representation of G on $\mathcal{H}_0$ such that

$$\pi(\alpha_t(a)) \;=\; U(t)\,\pi(a)\,U(t)^* \,, \qquad a \in \mathcal{A}\,,\; t \in G.$$

Given a non-degenerate representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_0)$ one can always construct a covariant representation on $L^2(G, \mathcal{H}_0)$, called the regular representation, by defining

$$\begin{aligned}(\pi^{\mathrm{reg}}(a)\psi)(t) &= \pi(\alpha_{-t}(a))\psi(t)\\ (U_s^{\mathrm{reg}}(a)\psi)(t) &= \psi(t-s).\end{aligned}$$

for all $a \in \mathcal{A}$, $s, t \in G$, $\psi \in L^2(G, \mathcal{H}_0)$.

The compactly supported norm-continuous functions $C_c(G, \mathcal{A})$ equipped with the operations

$$(fg)(t) \;=\; \int_G f(s)\,\alpha_s(g(t-s))\,\mathrm{d}s\,, \quad f^*(t) \;=\; \alpha_t(f(-t)^*)\,, \quad f, g \in C_c(G, \mathcal{A})\,, \tag{1.1.1}$$

are a $*$-algebra (for integration w.r.t. some Haar measure on G).

Given any covariant representation (π, U) one has an induced representation of $C_c(G, \mathcal{A})$ via the the integrated form

$$(\pi \times U)(f) = \int_G \pi(f(t))U(t)\mathrm{d}t.$$

The full crossed product $\mathcal{A} \rtimes_\alpha G$ is the C^*-algebra obtained by completion of $C_c(G, \mathcal{A})$ w.r.t. to the norm

$$\|f\| = \sup_{(\pi,U)} \left\|(\pi \times U)(f)\right\|$$

where the supremum is taken with respect to all covariant representations. The integrated form of any covariant representation extends uniquely to a representation of $\mathcal{A} \rtimes_\alpha G$, which is a universal property that almost determines that algebra up to isomorphism (see [105]). Since G is abelian one has the well-known result

Theorem 1.1.2 *If $\mathcal{A}$ acts faithfully and non-degenerately on a Hilbert space $\mathcal{H}_0$ then the induced representation $\pi^{\mathrm{reg}} \times U^{\mathrm{reg}}$ on $L^2(G, \mathcal{H}_0)$ is a faithful non-degenerate representation of $\mathcal{A} \rtimes_\alpha G$.*

On von Neumann algebras the notion of a strongly continuous action is too strong to be useful, instead one should require that the orbits under the action are continuous in one of the weaker operator topologies.

Definition 1.1.3 *A W^*-dynamical system is a triple $(\mathcal{M}, G, \alpha)$ consisting of a von Neumann algebra $\mathcal{M}$ and a weakly continuous action $\alpha : G \to \mathrm{Aut}(\mathcal{M})$, namely $t \in G \mapsto \alpha_t(a)$ is weak-$*$-continuous for any $a \in \mathcal{M}$. A covariant representation of a W^*-dynamical system is a pair (π, U) with π a non-degenerate normal representation of $\mathcal{M}$ on a Hilbert space $\mathcal{H}_0$ and U a strongly continuous unitary representation of G on $\mathcal{H}_0$, such that*

$$\pi(\alpha_t(a)) \;=\; U(t)\,\pi(a)\,U(t)^*\,, \qquad a \in \mathcal{M}\,,\;\; t \in G\,.$$

All of the usual weak operator topologies are equivalent here, e.g. one could equivalently ask for orbits continuous in the σ-strong or σ-weak operator topologies [92].

Definition 1.1.4 *Let $(\pi^{\mathrm{reg}}, U^{\mathrm{reg}})$ be a regular representation of a W^*-dynamical system $(\mathcal{M}, G, \alpha)$ on $\mathcal{H} = L^2(G, \mathcal{H}_0)$ constructed from a faithful non-degenerate* *normal representation $\pi : \mathcal{M} \to \mathcal{B}(\mathcal{H}_0)$. The W^*-crossed product $\mathcal{M} \rtimes_\alpha G$ is defined as the smallest von Neumann algebra in $\mathcal{B}(\mathcal{H})$ containing $\pi^{\mathrm{reg}}(\mathcal{M})$ and $U^{\mathrm{reg}}(G)$.*

A regular representation always exists and the definition does not depend on it up to isomorphism.

The dual group $\hat{G} = \mathrm{Hom}(G, S^1)$ acts canonically on a crossed product:

Definition 1.1.5 *Let $(\mathcal{B}, G, \alpha)$ be a C^*- or W^*-dynamical system with faithful regular representation (π, U). There is a strongly respectively weakly continuous action $\hat{\alpha}$ by $\hat{G}$ on $\mathcal{B} \rtimes_\alpha G$ where $\gamma \in \hat{G}$ acts in such a way that*

$$\hat{\alpha}_\gamma(\pi(a)\int_G f(t)U(t)\mathrm{d}t) = \pi(a)\int_G f(t)\gamma(t)U(t)\mathrm{d}t$$

for each $a \in \mathcal{B}$ and $f \in C_c(G)$.

Since the dual action has the right notion of continuity one can iterate the crossed product, which leads to a periodicity called Takai-duality in the C^*-algebraic case (see e.g. [105] for a simple proof) and Takesaki duality in W^*-algebraic case [120].

Theorem 1.1.6 *The iterated crossed product $\mathcal{A} \rtimes_\alpha G \rtimes_{\hat{\alpha}} \hat{G}$ of a C^*-dynamical system $(\mathcal{A}, G, \alpha)$ is isomorphic to $\mathcal{A} \otimes \mathbb{K}(L^2(G))$ where $\mathbb{K}(L^2(G))$ denotes the compact operators on $L^2(G)$. The isomorphism $i_T : \mathcal{A} \rtimes_\alpha G \rtimes_{\hat{\alpha}} \hat{G} \to \mathcal{A} \otimes \mathbb{K}(L^2(G))$ can be chosen in such a way that the second dual action $\hat{\hat{\alpha}}$ acts as $\alpha \otimes \mathrm{Ad}_{\rho_G}$, with ρ_G the regular representation of G on $L^2(G)$, i.e. acting by translation.*

In the same way the second crossed product $\mathcal{M} \rtimes_\alpha G \rtimes_{\hat{\alpha}} \hat{G}$ of a W^*-dynamical system is isomorphic to $\mathcal{M} \otimes \mathcal{B}(L^2(G))$.

1.2 Invariant traces and dual traces

In this section we recall some basic notions of traces on C^*-algebra and W^*-algebras. In combination with dynamical systems there are two important constructions, the first is the extension of invariant traces on a C^*-algebra to the von Neumann algebra generated by the associated GNS-representation, the other is the construction of a dual trace on a crossed product.

A weight ϕ on a C^*-algebra $\mathcal{A}$ is a convex-linear functional $\phi : \mathcal{A}^+ \to [0, \infty]$, where $\mathcal{A}^+$ is the cone of non-negative elements. A weight ϕ is called [92]

- a *trace* if $\phi(xx^*) = \phi(x^*x)$ for all $x \in \mathcal{A}$,
- *densely defined*, if $\phi(x) < \infty$ for a norm-dense subset of $\mathcal{A}^+$,
- *faithful*, if $\phi(x) = 0$ for $x \in \mathcal{A}^+$ if and only if $x = 0$.
- *lower semi-continuous*, if the sets $\{x \in \mathcal{A}^+ : \phi(x) \leq \lambda\}$ are norm-closed for all $\lambda \in \mathbb{R}_+$,
- *normal*, if $\phi(x) = \lim_{i\in I} \phi(x_i)$ whenever $x_i \uparrow x$ is an increasing net in $x \in \mathcal{A}^+$ with limit $x \in \mathcal{A}^+$.
- *semifinite*, if for every $x \in \mathcal{A}^+$ one has $\phi(x) = \sup_{\substack{0\leq y\leq x \\ \phi(y)<\infty}} \phi(y)$.

Traces on separable C^*-algebras often be required to be densely defined faithful and lower semi-continuous. On a von Neumann algebra traces are generally defined only on weakly dense domains and one needs weak continuity as well, therefore traces will instead to be required to be normal semifinite faithful (n.s.f.). A von Neumann algebra for which there exists a n.s.f. trace is called semifinite.

For a trace $\mathcal{T}$ define $\mathcal{A}^+_{\mathcal{T}}$ as the set of all positive elements $a \in \mathcal{A}^+$ such that $\mathcal{T}(a) < \infty$. The linear span of $\mathcal{A}^+_{\mathcal{T}}$ is denoted $\mathcal{A}_{\mathcal{T}}$ and is an ideal in $\mathcal{A}$. The trace $\mathcal{T}$ extends linearly to a tracial functional $\mathcal{T} : \mathcal{A}_{\mathcal{T}} \to \mathbb{C}$. The space of square-summable elements

$$\mathcal{A}^2_{\mathcal{T}} = \{a \in \mathcal{A} : \mathcal{T}(a^*a) < \infty\} ,$$

is also an ideal in $\mathcal{A}$ and by the polarization identity one has $\mathcal{T}(ab) = \mathcal{T}(ba)$ for all $a, b \in \mathcal{A}^2_{\mathcal{T}}$. Densely defined, faithful lower-semicontinuous traces extend to n.s.f. traces in GNS-representations and this is also compatible with group actions if the trace is invariant:

Proposition 1.2.1 ([111, Proposition 1.3.4]) *Let $(\mathcal{A}, G, \alpha)$ be a C^*-dynamical system and $\mathcal{T}$ be a densely defined, faithful and lower semi-continuous trace on $\mathcal{A}$. If $\mathcal{T}$ is α-invariant, namely*

$$\mathcal{T}(\alpha_t(a)) = \mathcal{T}(a) , \qquad \forall \ a \in \mathcal{A}^+_{\mathcal{T}} , \ t \in G ,$$

then $\mathcal{A}$ embeds covariantly into a von Neumann algebra denoted by $L^\infty(\mathcal{A},\mathcal{T})$ and α extends to a G-action $\tilde{\alpha}$ on this algebra such that $(L^\infty(\mathcal{A},\mathcal{T}), G, \tilde{\alpha})$ is a W^*-dynamical system with a n.s.f. trace on $L^\infty(\mathcal{A},\mathcal{T})$ which extends $\mathcal{T}$ and is invariant under $\tilde{\alpha}$.

Proof. We only recall the construction. The completion of $\mathcal{A}^2_{\mathcal{T}}$ w.r.t. to the a scalar product $(a|b) = \mathcal{T}(a^*b)$ is a Hilbert space $\mathcal{H}_{\mathcal{T}}$ on which $\mathcal{A}$ is represented via continuous extension of left multiplication $\pi_l : \mathcal{A} \to \mathcal{H}_{\mathcal{T}}$. This representation is called the semicyclic GNS-representation and is faithful for faithful $\mathcal{T}$. The action α extends from $\mathcal{A}^2_{\mathcal{T}}$ to a strongly continuous action on $\mathcal{H}$; it is implemented by a strongly continuous family of unitaries $(V(t))_{t\in G}$. One sets $L^\infty(\mathcal{A},\mathcal{T}) = \pi_l(\mathcal{A})''$ and extends α to a weakly continuous action by setting $\tilde{\alpha}_t = \mathrm{Ad}(V_t)$. The extension of the trace is constructed by noting that $\mathcal{A}^2_{\mathcal{T}}$ is a so-called Hilbert algebra (see Chapter I.6.2 of [46]) and therefore induces a n.s.f. trace on its algebra of bounded elements $L^\infty(\mathcal{A},\mathcal{T})$ which coincides with $\mathcal{T}$ on $\mathcal{A}_{\mathcal{T}} = \mathcal{A}^2_{\mathcal{T}}\mathcal{A}^2_{\mathcal{T}}$ and is $\tilde{\alpha}$-invariant since the scalar product is α-invariant. □

Knowing that the extension of the trace exists, it is determined uniquely through normality since $\mathcal{A}_{\mathcal{T}}$ is weakly dense.

The other important construction is that a dynamical system with invariant trace gives rise to a trace on a crossed product, called the dual trace. In view of the previous proposition it is enough to treat the case of W^*-dynamical systems, the dual trace for C^*-dynamical systems then is obtained by restricting to the C^*-algebraic crossed product.

Proposition 1.2.2 *Let $(\mathcal{M}, G, \alpha)$ be a W^*-dynamical system with an α-invariant n.s.f. trace $\mathcal{T}$ and fix a Haar measure μ on G. The crossed product $\mathcal{M} \rtimes_\alpha G$ has a n.s.f. trace $\hat{\mathcal{T}}_\alpha$, called the dual trace, which is invariant under α as well as the dual action $\hat{\alpha}$.*

The dual trace is uniquely determined by its values on a weakly dense subset, for which one can use the following prescription:

Assume that one can write $a_1, a_2 \in \mathcal{M}$ as SOT-integrals

$$a_i = \int_G \pi(f_i(t))U(t)\mathrm{d}\mu(t)$$

with compactly supported functions $f_i : G \to \mathcal{M}^2_{\mathcal{T}}$ *which are continuous w.r.t. the norm of* $L^2(\mathcal{M})$ *and the* σ*-*-strong operator topology, then* $a_1^* a_2$ *is* $\hat{\mathcal{T}}_\alpha$*-trace class and*

$$\hat{\mathcal{T}}_\alpha(a_1^* a_2) = \int_G \mathcal{T}(f_1(s)^* f_2(s)) \mathrm{d}\mu(s).$$

The construction is standard using Hilbert algebras (compare e.g. [122]) and the interplay with Takesaki duality works as expected:

Theorem 1.2.3 ([120]) *If* $\mathcal{T}$ *is an* α*-invariant n.s.f. trace on* $\mathcal{M}$*, then the second dual trace =* $(\hat{\mathcal{T}}_\alpha)^\wedge_{\hat{\alpha}}$ *on* $\mathcal{M} \rtimes_\alpha G \rtimes_{\hat{\alpha}} G$ *is carried by the isomorphism with* $\mathcal{M} \otimes \mathcal{B}(L^2(G))$ *into* $\mathcal{T} \otimes Tr$ *with Tr the usual trace on* $\mathcal{B}(L^2(G))$*.*

1.3 Non-commutative L^p-spaces

We recall the definition and the most important properties of the so-called non-commutative L^p-spaces (and refer to [127, 54, 98] for more detailed accounts). Let $\mathcal{M}$ be a von Neumann algebra with a n.s.f. trace $\mathcal{T}$ and acting on a Hilbert space $\mathcal{H}$. One can associate to $\mathcal{M}$, $\mathcal{T}$ linear metric L^p-spaces $L^p(\mathcal{M}, \mathcal{T})$, $0 < p < \infty$ by completing the appropriate subspaces of of $\mathcal{M}_{\mathcal{T}}$ in the obvious (quasi-)-norm. Instead of an abstract completion the elements of those spaces can also be realized as affiliated operators, which is a much more powerful construction [127, 54]. Recall that a densely defined unbounded operator T on $\mathcal{H}$ is called affiliated to $\mathcal{M}$ if each spectral projection of $|T|$ as well as the partial isometry of its polar decomposition are contained in $\mathcal{M}$.

A closed affiliated operator T is called $\mathcal{T}$-measurable if $\mathcal{T}(\chi(|T| > t)) < \infty$ holds for some large enough t [54]. It turns out that for any two measurable operators S, T the sum $S + T$ and product are ST are densely defined, closable and with the closure again measurable. In fact, the measurable operators form a topological $*$-algebra $\overline{\mathcal{M}}$ with strong sum and product and equipped with the measure topology, i.e. the topology generated by the neighborhood basis

$$N_{\epsilon,\delta}(x) \;=\; \{y \in \overline{\mathcal{M}} : \exists \text{projection } e \in \mathcal{M} \text{ with } \left\|(x-y)e\right\| < \epsilon, \mathcal{T}(1-e) < \delta\}\,.$$

The non-commutative L^p-spaces are then defined by

$$L^p(\mathcal{M}, \mathcal{T}) = \{T \in \overline{\mathcal{M}} : \|T\|_p := \mathcal{T}(|a|^p)^{\frac{1}{p}} < \infty\}.$$

As their classical counterparts they are (quasi-)Banach spaces and satisfy the Hölder inequality

$$\|ST\|_r \leq \|S\|_p \|T\|_q \tag{1.3.1}$$

for dual exponents $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and $0 < r, p, q \leq \infty$. In particular they satisfy the duality $L^p(\mathcal{M}, \mathcal{T})^* = L^q(\mathcal{M}, \mathcal{T})$ for $1 = \frac{1}{p} + \frac{1}{q}$ with $1 \leq p < \infty$. Moreover, they are interpolation spaces w.r.t. complex and real interpolation in the natural way [48], in particular this gives the log-convexity

$$\|S\|_r \leq \|S\|_p^{1-\theta} \|S\|_q^{\theta} \tag{1.3.2}$$

for $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$ for all $0 < \theta < 1$.

The trace $\mathcal{T}$ extends naturally and continuously to $L^1(\mathcal{M})$, i.e. $|\mathcal{T}(x)| \leq \|x\|_1$. The extension is a trace in the sense that

$$\mathcal{T}(xy) = \mathcal{T}(yx)$$

whenever $x \in L^p(\mathcal{M})$ and $y \in L^q(\mathcal{M})$ with $1 = \frac{1}{p} + \frac{1}{q}$.

The predual $\mathcal{M}_*$ can be identified with $L^1(\mathcal{M})$ since $x \in \mathcal{M} \mapsto \mathcal{T}(yx)$ is a normal functional for any $y \in L^1(\mathcal{M})$ and any normal functional has such a density with respect to $\mathcal{T}$.

For the non-commutative L^p-norms one has analogous convergence results to the Lemma of Fatou as well as of monotone and dominated convergence where the classical almost sure pointwise convergence is replaced by convergence in the measure topology [54].

Working with the measure topology is often impractical for us. Instead we want to interface the L^p-norms with convergence in one of the operator topologies. This is approached by noting that for sequences SOT-convergence is the same as convergence in the weak topology of $L^2(\mathcal{M})$.

Lemma 1.3.1 ([111, Lemma A.2.2]) *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}$ converging strongly to $a \in \mathcal{M}$ (and hence the sequence is uniformly bounded in norm).*

(i) Let $0 < p < \infty$. If $a_n \in \mathcal{M} \cap L^p(\mathcal{M})$ and $(a_n)_{n\in\mathbb{N}}$ converges in the L^p-(quasi-)norm to some $\tilde{a} \in \mathcal{M} \cap L^p(\mathcal{M})$, then $a = \tilde{a}$.

(ii) *Let* $1 < p < \infty$. *If* $\limsup_{n\to\infty} \|a_n\|_p < \infty$, *then* $(a_n)_{n\in\mathbb{N}}$ *converges in the weak* $\sigma(L^p(\mathcal{M}), L^q(\mathcal{M}))$*-topology where* $1 = \frac{1}{p} + \frac{1}{q}$.

(iii) *One has* $\|a\|_p \leq \liminf_{n\to\infty} \|a_n\|_p$ *for all* $1 < p < \infty$. *If in addition* $\text{s-lim}_{n\to\infty} a_n^* = a^*$, *then the same also holds for* $0 < p \leq 1$.

(iv) *For any sequence* $(b_n)_{n\in\mathbb{N}}$ *in* $L^p(\mathcal{M})$, $0 < p < \infty$ *converging in (quasi-)norm one has* $\lim_{n\to\infty} a_n b_n = ab_n$ *with convergence in* $L^p(\mathcal{M})$.

Another concept that generalizes to the non-commutative case is the L^p-ergodic theorem:

Theorem 1.3.2 ([133]) *Let* $\mathcal{M}$ *be a von Neumann algebra with n.s.f. trace* $\mathcal{T}$. *Let* $S : L^1(\mathcal{M}, \mathcal{T}) + \mathcal{M} \to L^1(\mathcal{M}) + \mathcal{M}$ *be a linear contractive map, i.e. satisfying* $\|S(x)\|_\infty \leq \|x\|_\infty$ *and* $\|S(x)\|_1 \leq \|x\|_1$, *and positive in the sense that* $S(x) \geq 0$ *for* $x \geq 0$. *Then the ergodic average*

$$\frac{1}{N}\sum_{k=0}^{N-1} S^k(x)$$

converges in L^p*-norm for every* $x \in L^p(\mathcal{M}, \mathcal{T})$ *and* $1 \leq p < \infty$.

Applying this to maps $S(x) := \frac{1}{T_0}\int_0^{T_0} \alpha_t(x)\mathrm{d}t$ for arbitrary $T_0 > 0$ an easy consequence is

Corollary 1.3.3 *Let* α *be a weakly continuous automorphic* $\mathbb{R}$*-action on* $\mathcal{M}$ *which leaves the n.s.f. trace* $\mathcal{T}$ *invariant. Then the extension to* $L^p(\mathcal{M}, \mathcal{T})$ *is strongly continuous, isometric and* α_t *is positive for any* $t \in \mathbb{R}$.

Hence the ergodic average

$$\langle x\rangle_T := \frac{1}{T}\int_0^T \alpha_t(x)\mathrm{d}t$$

converges in L^p*-norm to some* $\langle x\rangle_\infty \in L^p(\mathcal{M}, \mathcal{T})$ *for every* $x \in L^p(\mathcal{M})$ *and* $1 \leq p < \infty$. *Moreover the limit is invariant under the action* α_t.

1.4 Harmonic analysis

Given a Banach space with an isometric action of $\mathbb{R}^n$ it is natural to consider elements that are differentiable with respect to the action and introduce normed spaces for them. The precise construction must depend on the continuity of the action for which one can use the norm-topology or different weak or weak-$*$-topologies (see e.g. [33]). The relevant notions for this work are (for simplicity with $n = 1$)

- If E is a Banach space then the isometric action $\alpha : \mathbb{R} \times E \to E$ is called strongly continuous if the orbits under α are norm-continuous. An element $x \in E$ is called differentiable if

$$\lim_{\epsilon \to 0} \frac{\alpha_\epsilon(x) - x}{\epsilon}$$

  converges in norm. The smooth (infinitely often differentiable) elements are norm-dense [31].

- If $\mathcal{M}$ is a von Neumann algebra with weakly continuous automorphic action $\alpha : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ then an element $x \in \mathcal{M}$ is called weakly differentiable if there exists some element $y \in \mathcal{M}$ with

$$\psi(y) = \lim_{\epsilon \to 0} \psi\left(\frac{\alpha_\epsilon(x) - x}{\epsilon}\right)$$

  for each $\psi \in \mathcal{M}_*$. The weakly smooth elements are weakly dense in $\mathcal{M}$. If $\mathcal{M}$ acts on a Hilbert space $\mathcal{H}$ and α is generated by exponentiation of a self-adjoint operator D on $\mathcal{H}$ then $x \in \mathcal{M}$ is weakly differentiable if and only if x preserves the domain of D and the commutator $[D, x]$ extends to a bounded operator in $\mathcal{M}$ [32].

- If E is a Hilbert C^*-module or simply a Hilbert space and $\alpha : \mathbb{R} \times E \to E$ a strongly continuous action then the induced action on the bounded adjointable operators $\mathcal{B}(E)$ is continuous in the strict topology, i.e. the topology generated by the seminorms $\|xe\| + \|x^*e\|$ for each $e \in E$. An element $x \in \mathcal{B}(E)$ is called strictly differentiable if there is some $y \in \mathcal{B}(E)$ such that

$$\left(\frac{\alpha_\epsilon(x) - x}{\epsilon}\right) \to y$$

in the strict topology. Here α is extended to $\mathcal{B}(E)$ in the obvious manner $\alpha_t(x)e = \alpha_t(x\alpha_{-t}(e))$. The strictly smooth elements are strictly dense which follows from the usual smoothing trick just as in the cases above, e.g. one can approximate any multiplier $m \in \mathcal{B}(E)$ using the strictly smooth multipliers $\sum_{n=0}^{N} \widehat{W_n} * m$.

For an operator algebra $\mathcal{B}$ this gives different notions of derivative depending on whether $\mathcal{B}$ is a C^*-algebra with strongly continuous action, a von Neumann algebra with weakly continuous action or the multiplier algebra $M(\mathcal{A})$ of a C^*-algebra $\mathcal{A}$ with strongly continuous action (thus a Hilbert-$\mathcal{A}$-module).

1.4.1 Spaces of differentiable elements

Let $\alpha : \mathbb{R}^n \times E \to E$ be an isometric action on the Banach space E which is continuous in an appropriate sense as in the introduction (i.e. strongly, weakly or strictly continuous).

An element of E is called differentiable if for each $v \in \mathbb{R}^n$ the directional derivative

$$\nabla_v x = \frac{-1}{2\pi} \lim_{\epsilon \to 0} \frac{\alpha_{v\epsilon}(x) - x}{\epsilon} \tag{1.4.1}$$

converges in the appropriate sense. This gives a linear map $\nabla_v : \mathrm{Dom}(\nabla_v) \to E$ which is densely defined and closable in the respective topology. Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^d$ and set $\nabla_i := \nabla_{e_i}$ then the subspaces of the m-times differentiable elements is

$$C^m(E, \alpha) = \bigcap_{\substack{0 \le i_1, \dots, i_n \le m \\ \sum_k i_k \le m}} \mathrm{Dom}\left(\nabla_1^{i_1} \cdots \nabla_n^{i_n}\right),$$

and the subspace of smooth elements is

$$C^\infty(E, \alpha) = \bigcap_{m \in \mathbb{N}} C^m(E, \alpha).$$

Those subspaces are dense and $C^m(E, \alpha)$ is a Banach space in the norm

$$\|x\|_{\alpha,m} = \sum_{0 \le |j| \le m} \left\|\nabla^j x\right\|,$$

with multi-indices j and $\nabla^j = \nabla_{j_1} \cdots \nabla_{j_k}$ and supplied with the natural family of norms $C^\infty(E, \alpha)$ is a Fréchet space [31]. If E is a Banach algebra with strongly continuous action then ∇_i is a derivation and $C^m(E, \alpha)$ again a Banach algebra.

We will use this construction for several distinct cases:

- Let $(\mathcal{A}, \mathbb{R}^n, \alpha)$ be a C^*-algebra with densely defined lower-semicontinuous α-invariant trace $\mathcal{T}$. Then the trace-class elements $\mathcal{A}_\mathcal{T}$ are a Banach $*$-algebra with the norm [96]

$$\|a\|_\mathcal{T} \;=\; \|a\| \;+\; \mathcal{T}(|a|)\,. \tag{1.4.2}$$

  Since α restricted to $\mathcal{A}_\mathcal{T}$ is again strongly continuous one has a Fréchet algebra

$$\mathcal{A}_{\mathcal{T},\alpha} \;=\; C^\infty(\mathcal{A}_\mathcal{T}, \alpha)$$

  of smooth and summable elements w.r.t. $\|\cdot\|_\mathcal{T}$ is dense in $\mathcal{A}_\mathcal{T}$ w.r.t. $\|\cdot\|_\mathcal{T}$ and thus in particular norm-dense in $\mathcal{A}$.

- Let $(\mathcal{M}, G, \alpha)$ be a W^*-dynamical system where $\mathcal{M}$ carries an α-invariant n.s.f. trace $\mathcal{T}$. Then α extends uniquely to an isometric action $\alpha : G \times L^p(\mathcal{M}, \mathcal{T}) \to L^p(\mathcal{M}, \mathcal{T})$ which is strongly continuous w.r.t. the L^p-norm for $1 \le p < \infty$.

  In particular, the strong continuity implies that the smooth elements w.r.t. the action α are norm-dense in $L^p(\mathcal{M}, \mathcal{T})$. For $p \in [1, \infty)$ we define the non-commutative Sobolev spaces

$$W_p^m(\mathcal{M}, \mathcal{T}, \alpha) \;=\; C^m(L^p(\mathcal{M}, \mathcal{T}), \alpha)\,,$$

  with norms

$$\|a\|_{p,m} \;=\; \sum_{0 \le |j| \le m} \left\|\nabla^j x\right\|_p\,. \tag{1.4.3}$$

  Those spaces will often be denoted more briefly as $W_p^m(\mathcal{M}, \alpha)$ since any von Neumann algebra in this work has some distinguished trace; if the group action is also clear from the context we abbreviate even further to $W_p^m(\mathcal{M})$. Analogously one can construct the Fréchet space $W_p^\infty(\mathcal{M}, \alpha)$. The spaces $W_\infty^m(\mathcal{M}, \alpha)$ and $W_\infty^\infty(\mathcal{M}, \alpha)$ shall be the spaces of m times weakly differentiable and smooth elements respectively.

Notably, if $T \in L^p(\mathcal{M}, \mathcal{T})$ is an unbounded affiliated operator then it may have a well-defined derivative in the Sobolev sense. In a spatial representation one can show that this implies that the commutators of T with generators of the action $[X_i, T]$ are densely defined and measurable [44]. Another notion of differentiability for unbounded operator will be explored in Section 1.4.3 below.

1.4.2 Besov spaces

In some situations we may want to have finer graduations than merely the integer Sobolev spaces, but even fractional smoothness is not entirely sufficient; in the discussion of Hankel operators certain Besov spaces arise naturally. In this section we briefly recall the definition and properties of non-commutative Besov spaces on associated to semifinite von Neumann algebras based [111, Chapter 2].

In this section let $(\mathcal{M}, \alpha, G)$ be a W^*-dynamical system with n.s.f. α-invariant trace $\mathcal{T}$, where $G = \mathbb{T}^{n_0} \times \mathbb{R}^{n_1}$ with $n = n_0 + n_1$ is an n-parameter action. Let $p \in [1, \infty]$ be arbitrary and denote by $\|\cdot\|_p$ always the norm on $L^p(\mathcal{M}, \mathcal{T})$.

Denote the Fourier algebra by $FA(\mathbb{R}^n) = \mathcal{F}L^1(\mathbb{R}^n)$ which is with pointwise multiplication a subalgebra of $C_0(\mathbb{R}^n)$. For any $f \in FA(\mathbb{R}^n)$ a bounded operator on $L^p(\mathcal{M})$ is defined by

$$\alpha_f(x) = \int_G (\mathcal{F}^{-1} f)(t)\, \alpha_t(x)\, \mathrm{d}t\,, \qquad x \in E \tag{1.4.4}$$

in the sense of a Riemann integral for $p \in [1, \infty)$ and as a weak-$*$-convergent integral in the case of $p = \infty$. In either case, α_f defines a bounded map with

$$\left\|\alpha_f(x)\right\|_p \leq \left\|\hat{f}\right\|_1 \|x\|_p\,, \qquad \forall x \in L^p(\mathcal{M}),\ f \in FA(\mathbb{R}^n).$$

If the action α is clear from context then we prefer to write

$$\hat{f} * x := \alpha_f$$

to highlight its definition as a Fourier multiplier, which acts by convolution of its Fourier transform of f against the orbit of x. For fixed $x \in L^p(\mathcal{M})$ one obtains a representation of the Fourier algebra since

$$\widehat{f + \lambda g} * x = \hat{f} * x + \lambda \hat{g} * x, \qquad \widehat{fg} * x = \hat{g} * (\hat{f} * x) = \hat{f} * (\hat{g} * x)\,. \tag{1.4.5}$$

for the pointwise product on $FA(\mathbb{R}^n)$.

One can also define a notion of spectrum with respect to an action [12]:

Definition 1.4.1 *The Arveson spectrum of some $x \in L^p(\mathcal{M})$ is defined as*

$$\sigma_\alpha(x) \;=\; \{\lambda \in \mathbb{R}^n \;:\; f(\lambda) = 0 \;\; \text{for all } f \in FA(\mathbb{R}^n) \text{ with } \hat{f} * x = 0\}\,.$$

If x is in an intersection of two L^p-spaces it is not difficult to see that the two spectra coincide. When α is periodic then $\sigma_\alpha(x)$ is precisely the set of frequency components whose Fourier coefficients do not vanish.

Into the definition of the Besov spaces enters a smooth Littlewood-Payley decomposition consisting of Fourier multipliers from $FA(\mathbb{R}^n)$ [130, 93]. Let $\varphi : \mathbb{R} \to [0,1]$ be a smooth function with support $[-2,-2^{-1}] \cup [2^{-1},2]$ such that for all $x \in \mathbb{R} \setminus \{0\}$

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}x) \;=\; 1\,.$$

Choosing any such φ we fix one dyadic decomposition $(W_j)_{j\in\mathbb{N}}$ by

$$W_j(t) \;=\; \varphi(2^{-j}|t|) \quad \text{for } t \in \mathbb{R}^d,\; j>0\,, \qquad W_0 \;=\; 1 - \sum_{j>0} W_j \tag{1.4.6}$$

The support of each W_j for $j > 0$ is the annulus $\{t \in \mathbb{R}^n \;:\; 2^{j-1} \le |t| \le 2^{j+1}\}$ and the Fourier transforms of the W_j are an approximate unit for the convolution algebra since $\sum_{j=0}^{\infty} W_j(\lambda) = 1$. One can therefore show that

$$x \;=\; \sum_{j=0}^{\infty} \widehat{W_j} * x, \qquad \forall\, x \in L^p(\mathcal{M}) \tag{1.4.7}$$

with convergence in norm- for $p < \infty$ respectively in weak-$*$-topology for $p = \infty$.

Definition 1.4.2 *Given $q \in [1,\infty)$, $s > 0$ and a dyadic decomposition $(W_j)_{j\in\mathbb{N}}$ as above, the Besov norm of $x \in L^p(\mathcal{M})$ is defined by*

$$\|x\|_{B^s_{p,q}(\mathcal{M},\alpha)} \;=\; \Big(\sum_{j\ge 0} 2^{qsj}\, \|\widehat{W_j} * x\|_p^q\Big)^{\frac{1}{q}}, \quad \|x\|_{B^s_{p,\infty}(\mathcal{M},\alpha)} \;=\; \sup_{j\ge 0} 2^{sj}\, \big\|\widehat{W_j} * x\big\|_p\,.$$

The Besov space of scale s and q over $L^p(\mathcal{M})$ is then the Banach space

$$B^s_{p,q}(\mathcal{M},\alpha) \;=\; \left\{x \in L^p(\mathcal{M}) \;:\; \|x\|_{B^s_{p,q}(\mathcal{M},\alpha)} < \infty\right\}.$$

We conventionally suppress the dependence on α for readability. The definition is independent of the choice of dyadic decomposition, which is most easily seen from the characterization by differences, which eliminates it:

Theorem 1.4.3 ([111, Theorem 2.2.1.]) *Define the difference operator*

$$\Delta_t : L^p(\mathcal{M}) \to L^p(\mathcal{M}), \qquad \Delta_t(x) = \alpha_t(x) - x$$

and the N-th difference $\Delta^N_t = (\Delta_t \circ \ldots \circ \Delta_t)$. Then the N-th modulus of smoothness $\omega^N_p : L^p(\mathcal{M}) \times \mathbb{R}_> \to \mathbb{R}_\geq$ is

$$\omega^N_p(x,t) \;=\; \sup_{|r|\leq t} \left\|\Delta^N_r(x)\right\|_p \;. \tag{1.4.8}$$

For $q < \infty$ and any integer $N > s > 0$, the Besov norm $\|\cdot\|_{B^s_{p,q}(\mathcal{M})}$ is equivalent to the norm

$$\|x\|_{\widetilde{B}^s_{p,q}(\mathcal{M})} \;=\; \|x\|_p + \left(\int_{[0,1]} t^{-sq}\; \omega^N_p(x,t)^q \;\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} .$$

For $q = \infty$ and $N > s > 0$ it is equivalent to

$$\|x\|_{\widetilde{B}^s_{p,\infty}(\mathcal{M})} \;=\; \|x\|_p + \sup_{t\in[0,1]} \left(t^{-s}\; \omega^N_p(x,t)\right) .$$

There are some obvious inclusions between Besov space, based on the inequalities of the weighted sequence norms, namely $B^{s'}_{p,q'}(\mathcal{M}) \subset B^s_{p,q}(\mathcal{M})$ for $s \leq s'$ and $q \leq q'$. Likewise, interpolation also extends from the sequence norms:

Proposition 1.4.4 ([111, Proposition 2.1.3]) *For parameters $s_0, s_1 > 0$ and $1 \leq q_0, q_1 < \infty$, one has for the interpolation space of the Besov spaces of order $\theta \in (0,1)$*

$$\left(B^{s_0}_{p_0,q_0}(\mathcal{M}), B^{s_1}_{p_1,q_1}(\mathcal{M})\right)_\theta \;=\; B^s_{p,q}(\mathcal{M})\;,$$

where $s = (1-\theta)s_0 + \theta s_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and* $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. *Moreover, for* $x \in B^{s_0}_{p_0,q_0}(\mathcal{M}) \cap B^{s_1}_{p_1,q_1}(E_1)$,

$$\|x\|_{B^s_{p,q}(\mathcal{M})} \;\leq\; \|x\|^{1-\theta}_{B^{s_0}_{p_0,q_0}(\mathcal{M})} \|x\|^{\theta}_{B^{s_1}_{p_1,q_1}(\mathcal{M})} .$$

The Besov spaces are spaces of fractional smoothness in between the spaces of differentiable elements:

Proposition 1.4.5 ([111, Lemma 2.3.3-4]) *Let* $l \in \mathbb{N}$ *and* $x \in W^l_p(\mathcal{M}, \alpha)$ *(the Sobolev space of elements which are* l *times differentiable w.r.t.* α *in* p*-norm). For* $s < l$ *and* $q \in [1, \infty]$, *there is a constant* $C > 0$ *such that*

$$\|x\|_{B^s_{p,q}(\mathcal{M},\alpha)} \;\leq\; C \,\|x\|_{W^l_p(\mathcal{M},\alpha)} .$$

In particular there there is a constant $C > 0$ *such that for all* $j \geq 1$

$$\left\|\widehat{W}_j * x\right\|_p \;\leq\; C\, 2^{-jl} \,\|x\|_{W^l_p(\mathcal{M},\alpha)}$$

which shows that $W^n_p(\mathcal{M}) \subset B^s_{p,q}(\mathcal{M})$ *with continuous inclusion for each* $s > n$, $s \notin \mathbb{N}$ *and* $1 \leq q \leq \infty$.

Conversely, there exists for each multi-index m *a constant such that*

$$\left\|\nabla^m \widehat{W}_j * x\right\|_p \leq C 2^{|m|j} \left\|\widehat{W}_j * x\right\|_p$$

for all $j > 1$ *and thus* $B^s_{p,q}(\mathcal{M}) \subset W^n_p(\mathcal{M})$ *for all* $1 \leq q \leq \infty$ *and* $s < n$.

One also has a few non-standard inclusions that follow from interpolation of sequence norms:

Proposition 1.4.6 ([111, Lemma 2.3.5]) *(i) If* $a \in B^s_{p,p}(\mathcal{M})$ *is norm-bounded, i.e.* $a \in \mathcal{M}$, *then* $a \in B^{s\frac{p}{q}}_{q,q}(\mathcal{M})$ *for all* $q \geq p$ *with*

$$\|a\|_{B^{s\frac{p}{q}}_{q,q}} \;\leq\; \|a\|^{\frac{p}{q}}_{B^s_{p,p}} \,\|a\|^{1-\frac{p}{q}}_{\mathcal{M}} .$$

(ii) For any $p < q < p+1$ one has $W^1_q(\mathcal{M}) \cap \mathcal{M} \subset B^{\frac{p}{p+1}}_{p+1,p+1}(\mathcal{M}) \cap \mathcal{M}$.

The inclusion (ii) will be relevant in Chapter 2 since the Besov spaces of that form for integer p become important.

1.4.3 Differentiable multipliers

In this section let E be a Hilbert-C^*-module over a C^*-algebra $\mathcal{A}$. The results are for us only important in the two special cases $E = \mathcal{A}$ or where $E = \mathcal{H}$ is a Hilbert space, which one should keep in mind. We denote by $\mathcal{B}(E)$ the set of bounded adjointable $\mathcal{A}$-linear map supplied with strict topology, i.e. the topology generated by the semi-norms $\|ae\| + \|a^*e\|$ for all $e \in E$.

Recall that for $E = \mathcal{A}$ one has a canonical identification of $\mathcal{B}(E)$ with the multiplier algebra $M(\mathcal{A})$ and the strict topology is the same as the usual strict topology (also sometimes called the almost uniform topology, e.g in [132]). For $E = \mathcal{H}$ the strict topology is equal to the strong-$*$-operator topology.

Definition 1.4.7 *Let X be a densely defined regular self-adjoint operator on E, where all notions are understood in the Hilbert module sense. An operator $m \in \mathcal{B}(E)$ is called X-differentiable if*

(i) m and m^* preserve the domain $\mathcal{D}_X \subset E$ of X

(ii) $[X,m]$ and $[X,m^*]$ extend to bounded operators (which are then automatically adjointable).

Denote the set of all such operators by $\mathcal{B}_X(E)$.

Equivalently these are the operators such that the orbit under the one-parameter action generated by X is strictly differentiable:

Lemma 1.4.8 *In the situation of Definition 1.4.7 define the action*

$$\alpha : \mathbb{R} \times \mathcal{B}(E) \to \mathcal{B}(E), \quad \alpha_t(m) = e^{\iota X\cdot t} m e^{-\iota X \cdot t}.$$

One has $m \in \mathcal{B}_X(E)$ if and only if there exists an element $\nabla m \in \mathcal{B}_X(E)$ such that

$$\nabla m = \operatorname{s}^*\text{-}\lim_{t\to\infty} \frac{\alpha_t(m) - m}{t}$$

converges in the strict topology.

Proof. By the Hille-Yosida theorem the domain of X consists of precisely those $\phi \in E$ such that

$$\lim_{t\to 0} \frac{e^{\iota X t} - 1}{\iota t}\phi$$

converges in E, the limit defining $X\phi$.

For any $m \in \mathcal{B}(E)$ the orbit $t \in \mathbb{R} \mapsto \alpha_t(m)$ is strictly continuous. If m preserves the domain of X and $[X, m]$ extends to a bounded operator then

$$\begin{aligned}\lim_{t\to 0} \frac{(e^{\iota X t} m e^{-\iota X t} - m)\phi}{t} &= \lim_{t\to 0} \frac{e^{\iota X t}(m e^{-\iota X t}\phi - m\phi) + e^{\iota X t} m\phi - m\phi}{t} \\ &= -\iota m X\phi + \iota X m\phi = \iota[X, m]\end{aligned}$$

for each $\phi \in \mathrm{Dom}(X)$, hence the orbit is strictly differentiable at $t = 0$ and the derivative ∇m is the bounded extension of $\iota[X, m]$.

Assume for the other direction that the orbit is strictly differentiable. For $\phi \in \mathrm{Dom}(X)$ one has

$$\begin{aligned}\lim_{t\to 0} \frac{e^{\iota X t} - 1}{\iota t} m\phi &= \lim_{t\to 0} \frac{\alpha_t(m) e^{\iota X t}\phi - m\phi}{\iota t} \\ &= \lim_{t\to 0} \frac{\alpha_t(m) e^{\iota X t}\phi - \alpha_t(m)\phi + \alpha_t(m)\phi - m\phi}{\iota t} \\ &= mX\phi - \iota(\nabla m)\phi,\end{aligned}$$

hence $m\phi$ is also in the domain of X. The equality therefore reads $Xm\phi = mX\phi - \iota(\nabla m)\phi$, thus $\iota[X, m]$ extends to the bounded operator ∇m. □

The extension of strict differentiability to higher derivatives and $\mathbb{R}^d$-actions is obvious:

Definition 1.4.9 *Let $X_1, \dots, X_d$ be commuting densely defined regular self-adjoint operators on E with common core $\mathcal{D}_X^\infty = \bigcap_{k>0} \mathrm{Dom}(X_1^{k_1} \dots X_d^{k_d})$. For any operator m which preserves $\mathcal{D}_X^\infty$ define $\nabla_j(m) = \iota[X_j, m]$ and iteratively $\nabla^j(m)$ for any multi-index $j \in \mathbb{N}^d$ (which is then a multi-commutator).*

An operator $m \in \mathcal{B}(E)$ is called strictly (X-)smooth if

(i) m and m^* preserve $\mathcal{D}_X^\infty$.

(ii) $\nabla^j(m)$ *and* $\nabla^j(m^*)$ *extend to bounded operators for each multi-index* j *(and are thus adjoint to each other).*

Denote the set of all such smooth operators by $\mathcal{B}_X^\infty(E)$.

Actually the operators which are smooth in this strict sense are also smooth in an apparently stronger way:

Proposition 1.4.10 *In the situation of Definition 1.4.9 define the strictly continuous action*

$$\alpha : \mathbb{R}^d \times \mathcal{B}(E) \to \mathcal{B}(E), \quad \alpha_t(m) = e^{\imath X\cdot t} m e^{-\imath X\cdot t}.$$

One has

$$\mathcal{B}_X^\infty(E) = \{m \in \mathcal{B}(E) : t \in \mathbb{R}^d \mapsto \alpha_t(m) \text{ is norm-smooth}\}.$$

Proof. Norm-differentiability is clearly stronger than strict differentiability which gives the inclusion from right to left.

For the other inclusion it is enough to prove that any two times strictly differentiable function $t \in \mathbb{R} \mapsto f(t) \in \mathcal{B}(E)$ with bounded strict derivatives $f', f'' \in L^\infty(\mathbb{R}, \mathcal{B}(E))$ is norm-differentiable. By the fundamental theorem of calculus one has

$$(f(t) - f(s))\phi = \int_s^t f'(t)\phi \mathrm{d}t$$

for all $\phi \in E$ and hence

$$\|f(t) - f(s)\| \leq |s - t| \, \|f'\|_\infty$$

which proves that f is norm-continuous. Applying the same reasoning to f' instead of f one finds using boundedness of the second derivative that f' is also norm-continuous. One can therefore write

$$f(t) = f(0) + \int_0^t f'(t)\mathrm{d}t$$

as an operator-norm convergent Riemann integral (since the right-hand side converges and equals the left-hand side when evaluated on any element of E). This shows that f is norm-differentiable also with derivative f'. □

The argument also applies to other weak smoothness conditions, in particular the Sobolev spaces $W^{\infty}_{\infty}(\mathcal{M})$ defined in Section 1.4.1 are also composed of norm-smooth elements. The equivalence of strong and weak smoothness conditions is certainly known to experts (see e.g. [9, Corollary 5.A.3]), but surprisingly does not seem to come up often in the literature compared to the equivalence of strong and weak analyticity.

The remainder of this section covers smooth unbounded self-adjoint operators. Obviously there is no single notion of smoothness of an unbounded operator H that encompasses all use cases, instead there are many possible definitions of various strengths. For example one could simply require that some bounded functions of H such as the resolvents $(H+z)^{-1}$ or the bounded transform $F(H)$ are smooth. This would not be enough, however, since a main interest in applications is often control of the (possibly unbounded) commutators $[X_i, H]$ which means that one must also impose domain conditions on H itself to assert the well-definedness of such commutators. Also the notion should be stable under addition of smooth bounded self-adjoint perturbations. A well-known criterion, based on smoothness of the resolvent is given in [9], however, that is not strong enough for our purposes since it does not imply smoothness of switch functions, such as the bounded transform $F(H)$. In a recent work [112] a definition for differentiability was proposed based on certain compatibility conditions introduced in [64] that are important in the context of the Kasparov product and which transfers immediately to differentiability of the bounded transform. This section presents an attempt to give a notion of infinitely often differentiable elements that is similar in spirit.

Definition 1.4.11 *Let H be a regular self-adjoint operator on E and $X = X_1, ..., X_d$ be the commuting regular self-adjoint operators on E with common core $\mathcal{E}_X$ that is dense in $\mathcal{D}^{\infty}_X$ w.r.t. the Fréchet topology generated by the graph norms of all X^j, $j \in \mathbb{N}^d$.*

We say that H is strictly (X-)smooth if

(i) *There are the inclusions*

$$\begin{aligned} (H + \imath\mu)^{-1}\mathcal{E}_X &\subset \mathrm{Dom}(H) \cap \mathcal{E}_X, \\ X^j(H + \imath\mu)^{-1}\mathcal{E}_X &\subset \mathrm{Dom}(H) \cap \mathcal{E}_X, \\ HX^j(H + \imath\mu)^{-1}\mathcal{E}_X &\subset \mathcal{E}_X \end{aligned}$$

for all multi-indices $j \in \mathbb{N}^d$ and all $\mu \neq 0$.

(ii) *The densely defined symmetric operators $\nabla^j H$ are relatively bounded w.r.t. H in the sense that*

$$(\nabla^j H)(H + \imath\mu)^{-1}$$

extends from $\mathcal{E}_X$ to a bounded operator for any (and thus all) $\mu \neq 0$ and all multi-indices j.

(iii) *There shall be some $0 < \eta < \frac{1}{2}$ such that $(\nabla^j H)(1 + H^2)^{-\eta}$ extends from $(1 + H^2)^{-1+\eta}\mathcal{E}_X$ to a bounded operator.*

To control the size of the commutators we also introduce the quantities

$$|||H|||_{k,\eta} = \sup_{|j|=k} \left\|(\nabla^j H)(1 + H^2)^{-\eta}\right\|.$$

If the value of η is important we may speak of (X, η)-smoothness.

Due to (i), the operator $(\nabla^j H)(H + \imath\mu)^{-1}$ is well-defined initially on the dense subspace $\mathcal{E}_X$ and the multi-commutator can be completely expanded into a sum of terms $X^{j_1} H X^{j_2}$ without domain issues. The condition (iii) is apart from domain considerations strictly stronger than (ii); if H is a polynomial function of degree n and X a first-order differential operator then one will try to use the criterion for the fraction $\eta = \frac{n-1}{2n}$ (generically $\nabla^j H$ could even be bounded relative to $|H|^{\frac{n-j}{n}}$ but choosing the power independent of j is sufficient for our purposes).

The following rather technical proof is inspired by techniques from [64]:

Lemma 1.4.12 *Let H be strictly X-smooth.*

(i) *If H is strictly X-smooth with the conditions Definition 1.4.11(i-iii) holding for some core $\mathcal{E}_X$ then they automatically hold for $\mathcal{E}_X = \mathcal{D}_X^\infty$ as well.*

(ii) *$\mathcal{D}_H = (H + \imath)^{-1}\mathcal{D}_X^\infty$ is a core simultaneously for H and all X^j, which satisfies $X^j \mathcal{D}_H \subset \mathcal{D}_H$.*

(iii) *The maps $(H - \imath\mu)^{-1}\nabla^j(H)$ and $(1 + H^2)^{-\eta}\nabla^j(H)$ extend from $\mathcal{D}_H$ to bounded operators which are then respectively the adjoints of $\nabla^j(H)(H + \imath\mu)^{-1}$ and $\nabla^j(H)(1 + H^2)^{-\eta}$.*

Proof. For the point *(i)* one only needs to prove the domain inclusions of 1.4.11(i), since the bounded extensions of 1.4.11(ii) and 1.4.11(iii) are unique. Let $\psi \in \mathcal{D}_X^\infty$ be arbitrary, then there exists, by assumption on $\mathcal{E}_X$, a sequence $(\psi_n)_{n\in\mathbb{N}}$ in $\mathcal{E}_X$ such that $X^j\psi_n \to \psi$ for all j. To prove that one can replace $\mathcal{E}_X$ with $\mathcal{D}_X^\infty$ it is enough to show that for $n \to \infty$ one has convergence w.r.t. various graph norms, concretely we need to show that in the norm of E one has

$$\begin{aligned}
(H + \imath\mu)^{-1}\psi_n &\to (H + \imath\mu)^{-1}\psi \\
H(H + \imath\mu)^{-1}\psi_n &\to H(H + \imath\mu)^{-1}\psi \\
X^j(H + \imath\mu)^{-1}\psi_n &\to X^j(H + \imath\mu)^{-1}\psi \\
HX^j(H + \imath\mu)^{-1}\psi_n &\to HX^j(H + \imath\mu)^{-1}\psi \\
X^{j_1}HX^{j_2}(H + \imath\mu)^{-1}\psi_n &\to X^{j_1}HX^{j_2}(H + \imath\mu)^{-1}\psi
\end{aligned}$$

for all multi-indices j, j_1, j_2. Since H and X^j are closed it is sufficient to show that the limits do exist at all. The first two are obvious and for the third we note that one can expand

$$X^j(H + \imath\mu)^{-1}\phi_n = \sum_{k\leq j}(H + \imath\mu)^{-1}B_k^{(j)}X^k\phi_n$$

for some bounded operators B_k by iterating

$$X_i(H + \imath\mu)^{-1}\psi_n = (H + \imath\mu)^{-1}(X_i\psi_n) + (H + \imath\mu)^{-1}[X_i, H](H + \imath\mu)^{-1}\psi_n \quad (1.4.9)$$

and using the boundedness of the operators in 1.4.11(ii). More precisely, each $B_k^{(j)}$ is a product of operators $\nabla^m(H)(H + \imath\mu)^{-1}$ for different m and we never run into domain issues due to $\phi_n \in \mathcal{E}_X$. To reduce the convergence of the remaining sequences to convergence in graph norm one similarly expands

$$HX^j\phi_n = \sum_{k\leq j} H(H + \imath\mu)^{-1}B_k^{(j)}X^k\phi_n$$

and finally

$$\begin{aligned}
X^{j_1}HX^{j_2}\phi_n &= X^{j_1}\sum_{k\leq j_2}(1 - \imath\mu(H + \imath\mu)^{-1}B_k^{(j_2})X^k\phi_n \\
&= \sum_{k\leq j_2} X^{j_1+k}\phi_n - \imath\mu\sum_{k\leq j_2} X^{j_1}(H + \imath\mu)^{-1}B_k^{(j_2}X^k\phi_n \\
&= \sum_{k\leq j_2} X^{j_1+k}\phi_n - \imath\mu(H + \imath\mu)^{-1}\sum_{m\leq j_1}\sum_{k\leq j_2} B_m^{(j_1)}X^{j_1}B_k^{(j_2)}X^k\phi_n
\end{aligned}$$

$$= \sum_{k \le j_2} X^{j_1+k}\phi_n - \imath(H+\imath\mu)^{-1} \sum_{l \le j_1+j_2} C_l^{(j_1,j_2)} X^l \phi_n$$

with some bounded operators $C_l^{(j_1,j_2)}$ since $[X_i, B_k^{(j_2)}]$ is again a bounded operator (it can again be written as a sum of products with factors $\nabla^m(H)(H+\imath\mu)^{-1}$). This completes the proof of (i).

Clearly $\mathcal{D}_H := (H+\imath)^{-1}\mathcal{D}_X^\infty$ is a core for H since each $\psi \in \mathrm{Dom}(H)$ can be written in the form $\psi = (H+\imath)^{-1}\varphi$ and φ can be approximated by a sequence in $\mathcal{D}_X^\infty$. From (i) it follows that $\mathcal{D}_H$ is invariant under all X^j.

The statement (iii) about the adjoints then follows easily; for all $\varphi \in \mathcal{D}_H$ and $\phi = (H+\imath)^{-1}\psi \in \mathcal{D}_H$ one clearly has

$$\langle \phi, \nabla^j(H)(H+\imath\mu)^{-1}\varphi\rangle_E = \langle (H-\imath\mu)^{-1}\nabla^j(H)(H+\imath)^{-1}\psi, \varphi\rangle_E$$

since the multi-commutator can be completely expanded without domain issues. Hence one has exhibited that $(H-\imath\mu)^{-1}\nabla^j(H)$ is a densely defined adjoint for $\nabla^j(H)(H+\imath\mu)^{-1}$, since the latter is bounded the former must be as well.

It remains to show the rest of (ii), namely that $\mathcal{D}_H$ is a core for all X^j, which is done by defining for each $\psi \in \mathcal{D}_H$ the approximating sequence

$$\psi_n = (1 + \frac{\imath}{n}H)^{-1}\psi, \qquad \in \mathcal{D}_H$$

and showing that $X^j\psi_n \to X^j\psi$ for all j.
For $j = e_k$ this follows from the strict convergence $(1+\frac{\imath}{n}H)^{-1} \to \mathbb{1}$ and the fact that $[X_k, (1+\frac{\imath}{n}H)^{-1}]$ converges to 0 in the strict topology due to

$$\begin{aligned}(\nabla_k(1+\frac{\imath}{n}H)^{-1})\psi &= \frac{\imath}{n}(1+\frac{\imath}{n}H)^{-1}\nabla_k(H)(1+\frac{\imath}{n}H)^{-1} \\ &= \frac{\imath}{n}(1+\frac{\imath}{n}H)^{-1}\nabla_k(H)(\imath+H)^{-1}(\imath+H)(1+\frac{\imath}{n}H)^{-1}\psi\end{aligned}$$

which converges strictly to 0 since $\frac{\imath}{n}(\imath+H)(1+\frac{\imath}{n}H)^{-1} \to \mathbb{1}$ strictly and

$$\sup_{n>0}\left\|(1+\frac{\imath}{n}H)^{-1}\nabla_k(H)(\imath+H)^{-1}\right\| < \infty.$$

For larger j one can iterate the argument to expand

$$X^j(1+\frac{\iota}{n}H)^{-1}\psi = \sum_{k\le j} B_{n,k}X^k\psi$$

with bounded operators $B_{n,k}$ that converge to 0 strictly. the argument; expanding $X^j(1+\frac{\iota}{n}H)^{-1}$ one obtains a sum of terms where one can always bracket out $(\iota+H)(1+\frac{\iota}{n}H)^{-1}\psi$ on the right and the remaining terms factors are uniformly bounded in n. □

One of our main interests was to find a criterion for bounded functions of an unbounded multiplier to be smooth. It is clear that the resolvent of a strictly smooth multiplier is always smooth and using the smooth functional calculus (see Appendix A) it is possible to write any $f(H)$ for a smooth and rapidly decaying function f as an integral in terms of the resolvent, hence those functions are strictly smooth as well. While that integral formula does not generally converge for functions f that do not decay at infinity, one can use approximation to extend the range of applicability to functions in a class $f\in\mathcal{S}^\beta(\mathbb{R})$, i.e. those functions whose k-th derivatives roughly decay faster than $\langle x\rangle^{-\beta-k}$ (see Definition A.1).

Lemma 1.4.13 *Let H be a strictly smooth operator for some $0\le\eta<\frac{1}{2}$ and let $f\in\mathcal{S}^\beta(\mathbb{R})$ be bounded with $\beta>-1+2\eta$. Then $f(H)$ preserves $\mathcal{D}_X^\infty$ and all $\nabla^j f(H)$ extend to bounded operators. More quantitatively, there is for each $j>0$ a constant c_j, independent of H and f, such that*

$$\left\|\nabla^j f(H)\right\| \le c_j \|f\|_{\mathcal{S}_{|j|}^{-(1-2\eta)}} \sup_{k>0} \sup_{\substack{m_1,\dots,m_k\ge1\\ \sum_i m_i=j}} \prod_{i=1}^k |||H|||_{m_i,\frac{1}{2}}. \tag{1.4.10}$$

Proof. By approximation it will be enough to prove the statement for $f\in C_c^\infty(\mathbb{R})$ as long as uniform norm-bounds are established. Let $\psi\in\mathcal{D}_H$ and let $\tilde{f}_K$ be a an almost analytic extension for f such that (see Appendix A)

$$f(H)\psi = \int_{\mathbb{C}} (\partial_{\overline{z}}\tilde{f}_K)(z)(H+z)^{-1}\psi\,\mathrm{d}z\wedge\mathrm{d}\overline{z}.$$

Since $\left\|(H+z)^{-1}(H+\iota)^{-1}\right\|\le c\,|\Im m\,z|^{-1}\,\langle\Re e\,z\rangle^{-1}$ the integral on the right converges absolutely in the norm of E (since we assume that f is compactly supported, otherwise that only holds for $\beta>0$).

By expanding $(H+z)^{-1}$ into a norm-convergent series around some $(H+\imath\mu)^{-1}$ with $|\mu|$ large enough one can verify that $(H+z)^{-1}$ also preserves $\mathcal{D}_X^\infty$ and $\nabla^j(H)(H+z)^{-1}$ extends to a bounded operator. Due to similar commutator algebra as in the proof of Lemma 1.4.12 one then finds that the function

$$z \in \mathbb{C}\setminus\mathbb{R} \mapsto \nabla^j(H)(H+z)^{-1}\phi$$

is continuous in the graph norm of any X^k for $\phi \in \mathcal{D}_X^\infty$. Since X^j is closed that means one can show that $f(H)$ preserves the domain of X^j by proving that

$$\int_{\mathbb{C}} (\partial_{\overline{z}}\tilde{f}_K)(z) X^j (H+z)^{-1}\phi \,\mathrm{d}z \wedge \mathrm{d}\overline{z}$$

converges absolutely in E for any $\phi \in \mathcal{D}_X^\infty$. Likewise, knowing that $f(H)$ preserves $\mathcal{D}_X^\infty$ and that

$$\int_{\mathbb{C}} (\partial_{\overline{z}}\tilde{f}_K)(z) \nabla^j (H+z)^{-1} \,\mathrm{d}z \wedge \mathrm{d}\overline{z}$$

converges in operator norm the integral must be a bounded extension of $\nabla^j f(H)$.

The estimates for both scenarios work the same, via the Leibniz identity for commutators we can expand terms like $X^j(H+z)^{-1}\phi$ or $\nabla^j(H+z)^{-1}\phi$ as a sum of terms of the form

$$(H+z)^{-1}\prod_{m=1}^{k}\left((\nabla^{j_m}H)(H+z)^{-1}\right)X^{j_{k+1}}\phi \tag{1.4.11}$$

with multi-indices j_m such that $\sum_{m=1}^{k+1} j_m = j$ and $1 \leq k \leq |j|$. One then estimates

$$\begin{aligned}
&\|(1.4.11)\| \\
&\leq \left\|(H+z)^{-1}\right\| \prod_{m=1}^{k} \left\|(\nabla^{j_m}H)(1+H^2)^{-\eta}\right\| \left\|(1+H^2)^{\eta}(H+z)^{-1}\right\| \left\|X^{j_{k+1}}\phi\right\| \\
&\leq |\Im m\, z|^{-1-k(1-2\eta)}\left(1+\frac{\langle \Re e\, z\rangle}{|\Im m z|}\right)^{2k\eta} \prod_{m=1}^{k} |||H|||_{j_m,\eta} \left\|X^{j_{k+1}}\phi\right\|.
\end{aligned}$$

In particular, taking all allowed combinations into account (those with $j_{k+1} = 0$) one has

$$\left\|\nabla^j (H+z)^{-1}\right\| \leq \sum_{m=1}^{|j|} C_m \, |\Im m\, z|^{-1-m(1-2\eta)} \left(1 + \frac{\langle \Re e\, z\rangle}{|\Im m z|}\right)^{2m\eta} \sup_{k>0} \sup_{\substack{m_1,\dots,m_k \geq 1 \\ \sum_i m_i = j}} \prod_{i=1}^{k} |||H|||_{m_i,\eta}$$

for some universal constants C_m. From Lemma A.4 one therefore finds that for $f \in \mathcal{S}^\beta(\mathbb{R})$ that

$$\int_{\mathbb{C}} \left|(\partial_{\bar z} \tilde f_K)(z)\right| \left\|\nabla^j (H+z)^{-1}\right\| \, \mathrm{d}z \wedge \mathrm{d}\bar z \leq c_{j,s,K} \, \|f\|_{\mathcal{S}_K^{-(1-2\eta)}}$$

if $K > r + s$ and $\beta \geq -1 - 2\eta$. As argued above, that shows that $f(H)$ preserves $\mathcal{D}_X^\infty$ and gives the desired norm bound (1.4.10).

Now we need to extend this to functions without compact support. The argument above actually produces a bound

$$\left\|X^j f(H)\phi\right\|_E \leq c_j \sum_{k \leq j} \|f\|_{\mathcal{S}_{|j|}^{-(1-2\eta)}} \left\|X^k \phi\right\|$$

which implies that any $f(H)$ with $f \in \mathcal{S}^{-(1-2\eta)}(\mathbb{R})$ preserves $\mathcal{D}_X^\infty$, since there is a sequence compactly supported functions with $f_n(H) \to f(H)$ strictly and all X^j are closed. Likewise the commutator is bounded by continuity w.r.t. f. □

Let us highlight the following special case:

Corollary 1.4.14 *The bounded transform $F(H) = H(1+H^2)^{-\frac{1}{2}}$ of a strictly X-smooth Hamiltonian is strictly X-smooth.*

Proof. The function F lies in $\bigcap_{0>\beta>-1} \mathcal{S}^\beta(\mathbb{R})$, hence $F(H)$ satisfies the conditions of the Lemma for any $0 < \eta < \frac{1}{2}$. □

The following result is sometimes also useful in functional calculus:

Lemma 1.4.15 *If H is strictly X-smooth for some $0 < \eta < \frac{1}{2}$ then the bounded extensions of*

$$(\nabla^j H)(H + \imath)^{-1}$$

and

$$(\nabla^j H) f(H)$$

for $f \in \mathcal{S}^\beta(\mathbb{R})$ with $\beta \geq 2\eta$ are strictly smooth operators with derivatives bounded in terms of $\|f\|_{\mathcal{S}^{2\eta}_{|j|}}$.

Proof. It is clear that $(\nabla^j H)(H + \imath)^{-1}$ preserves $\mathcal{D}_X^\infty$ and additional derivatives can again be expanded via the Leibniz identity to bounded operators. To see that the same holds for the adjoint we note that $(H + \imath)^{-1}\nabla^j H$ preserves the core $\mathcal{D}_H$ and each $\nabla^k((H + \imath)^{-1}\nabla^j H)$ extends from there to a bounded operator. Therefore the bounded extension of $(H + \imath)^{-1}\nabla^j H$ actually does preserve $\mathcal{D}_X^\infty$ by a similar argument as in Lemma 1.4.12.

To check the smoothness of $\nabla^j(H)f(H)$ we note that one has for compactly supported f a norm-convergent integral

$$(\nabla^j H) f(H) = \int_{\mathbb{C}} (\partial_{\overline{z}} \tilde{f}_K)(z)(\nabla^j H)(H + z)^{-1}\psi \,\mathrm{d}z \wedge \mathrm{d}\overline{z}$$

for all $\psi \in \mathcal{D}_X^\infty$ and $\tilde{f}_K$ a quasi-analytic extension. Since one can bound

$$\left\|(\nabla^j H)(H + z)^{-1}\right\| \leq |\Im m\, z|^{-1+2\eta} \left(1 + \frac{\langle \Re e\, z\rangle}{|\Im m z|}\right)^{2\eta}$$

one has, again using Lemma A.4, a uniform estimate $\left\|(\nabla^j H) f(H)\right\| \leq c\,\|f\|_{\mathcal{S}^{2\eta}_K}$ for each $K > 1$ and similarly for derivatives which allows one to remove the condition of compact support.

One can now follow the same strategy as in the proof of Proposition 1.4.13 and show that the integral converges in all graph norms, hence $\nabla^j(H)f(H)$ preserves $\mathcal{D}_X^\infty$, and also differentiate under the integral sign using the Leibniz identity which gives norm-convergent expressions for $\nabla^k(\nabla^j(H)f(H))$. For the adjoints one argues similarly that $f(H)\nabla^j(H)$ preserves the core $\mathcal{D}_H$ and the $\nabla^k(f(H)\nabla^j(H))$ extend from there to bounded operators starting from the expression

$$f(H)\nabla^j(H)(H + \imath)^{-1} = \int_{\mathbb{C}} (\partial_{\overline{z}} \tilde{f}_K)(z)(H + z)^{-1}\nabla^j(H)(H + \imath)^{-1}\psi \,\mathrm{d}z \wedge \mathrm{d}\overline{z}.$$

□

Perturbations can be handled similarly:

Proposition 1.4.16 *Let H be a regular self-adjoint strictly smooth operator and V a bounded self-adjoint strictly smooth operator. Then $H+V$ is again a regular self-adjoint X-smooth operator.*

Proof. The sum is regular self-adjoint by Kato-Rellich (see [64] for the Hilbert module case). Since $\mathrm{Dom}(H+V) = \mathrm{Dom}(H)$ and

$$(H+V+\imath\mu)^{-1} = (H+\imath\mu)^{-1} - (H+\imath\mu)^{-1}V(H+V+\imath\mu)^{-1}, \tag{1.4.12}$$

the domain inclusions of (i) will follow immediately if one can assert that the bounded operator $V(H+V+\imath\mu)^{-1}$ preserves $\mathcal{D}_X^\infty$. Let initially be $|\mu|$ so large that $\left\|V(H+V+\imath\mu)^{-1}\right\| < 1$, then we write as a norm-convergent sum

$$V(H+V+\imath\mu)^{-1} = \sum_{m=0}^{\infty}(-1)^m\left(V(H+\imath\mu)^{-1}\right)^{m+1} =: \sum_{m=0}^{\infty} A^{m+1}. \tag{1.4.13}$$

The bounded operator A is smooth and due to the Leibniz identity one estimates $\left\|\nabla^k A^m\right\| \leq c_k \left\|A\right\|^{m-|k|}$ with $\|A\| < 1$ by assumption. Since

$$X^j A^m = \sum_{k\leq j} c_{k,j}\nabla^k(A^m)X^{j-k}$$

for some universal constants $c_{k,j}$ one sees that (1.4.13) maps any sequence converging in the Fréchet topology of $\mathcal{D}_X^\infty$ to a sequence that converges in the graph norm of any X^j, thus $(H+V+\imath\mu)^{-1}$ preserves $\mathcal{D}_X^\infty$ for large enough μ. Then (1.4.12) implies that $\nabla^j(H+V)(H+V+\imath\mu)^{-1}$ is well-defined and extends to a bounded operator for any multi-index j. By iteratively expanding $(H+V+\imath\mu)^{-1}$ around μ one eventually concludes from this that any resolvent $(H+V+\imath\mu)^{-1}$ for $|\mu| > 0$ preserves $\mathcal{D}_X^\infty$ and any $\nabla^j(H+V)(H+V+\imath\mu)^{-1}$ extends to a bounded operator. It remains to verify condition (iii), which is equivalent to providing an estimate

$$\left\|\nabla^j(H+V)(H+V+\imath)^{-1}\phi\right\| \leq c\left\|(1+(H+V)^2)^\eta(H+V+\imath)^{-1}\phi\right\|$$

valid for all $\phi \in \mathcal{D}_X^\infty$. Inserting a 1 we can bound

$$\|\nabla^j(H+V)(H+V+\imath)^{-1}\phi\| \\ \leq \left\|\nabla^j(H+V)(1+H^2)^{-\eta}\right\| \left\|(1+H^2)^{\eta}(H+V+\imath)^{-1}\phi\right\|$$

and inserting another 1 in the second norm on the right-hand side is bounded by by

$$\left\|(1+H^2)^{\eta}(1+(H+V)^2)^{-\eta}\right\| \left\|(1+(H+V)^2)^{\eta}(H+V+\imath)^{-1}\phi\right\|.$$

The second factor has bounded norm due to functional calculus and to see that the norm of the first factor is also finite we use the smooth functional calculus to write

$$(1+(H+V)^2)^{-\eta} = (1+H^2)^{-\eta} + (1+H^2)^{-\eta}B$$

with the bounded operator

$$B = \frac{1}{2\pi}\int_{\mathbb{C}} (\partial_{\overline{z}}\tilde{f}_K)(z)(1+H^2)^{\eta}(H+z)^{-1}V(H+V+z)^{-1}\,\mathrm{d}z\wedge\mathrm{d}\overline{z}$$

for $\tilde{f}_K$ an almost analytic extension of $\lambda \mapsto (1+\lambda^2)^{-\eta}$. Hence

$$(1+(H+V)^2)^{-\eta}E \subset (1+H^2)^{-\eta}E \subset \mathrm{Dom}((1+H^2)^{\eta})$$

and $(1+H^2)^{\eta}(1+(H+V)^2)^{-\eta}$ extends to a bounded operator. □

The boundedness of V was actually not used very much, it could be replaced by the conditions that $V(H+\imath\mu)^{-1}$ shall be bounded and smooth, as well as that $V(1+H^2)^{-\gamma}$, $0<\gamma<1$ and $\nabla^j(V)(1+H^2)^{-\eta}$ extend to bounded operators.

One can also do more quantitative estimates; for that it is useful to simultaneously do estimates in other operator-ideal norms:

Definition 1.4.17 *Let $\mathcal{C}$ be a norm-closed $*$-subalgebra of $\mathcal{B}(E)$ and $\mathcal{J}$ a (not necessarily closed) ideal in $\mathcal{C}$. We say that $\mathcal{J}$ is a strictly closed operator ideal if $\mathcal{J}$ is a Banach-$*$-algebra in some norm $\|\cdot\|_{\mathcal{J}}$ such that*

(i) *The inclusion $\mathcal{J} \hookrightarrow \mathcal{C}$ is continuous.*

(ii) $\|c_1 x c_2\|_{\mathcal{J}} \leq \|c_1\| \; \|j\|_{\mathcal{J}} \; \|c_2\|$ *for all $x \in \mathcal{J}$ and $c_1, c_2 \in \mathcal{C}$.*

(iii) *If a sequence x_n in $\mathcal{J}$ converges strictly to some $c \in \mathcal{C}$ with uniformly bounded $\mathcal{J}$-norm, then $c \in \mathcal{J}$ with*

$$\|c\|_{\mathcal{J}} \leq \liminf_{n\to\infty} \|x_n\|_{\mathcal{J}}.$$

Both conditions hold for $\|\cdot\|_{\mathcal{J}}$ the operator norm but also for e.g. non-commutative L^p-norms $\|\cdot\|_\infty + \|\cdot\|_p$ (the condition (iii) is more or less the Lemma of Fatou, compare Lemma 1.3.1).

Definition 1.4.18 *Let $\mathcal{J}$ be a strictly closed operator ideal in $\mathcal{C} \subset \mathcal{B}(E)$.*

Let H be self-adjoint strictly smooth multiplier with

$$(H+z)^{-1} \in \mathcal{C}, \quad \nabla^j(H)(H+z)^{-1} \in \mathcal{C}, \quad \nabla^j(H)(1+H^2)^{-\eta} \in \mathcal{C}$$

for each z in the resolvent set of H.

We say that a strictly smooth bounded operator is a $\mathcal{J}$-smooth perturbation of H if there is some $0 < \gamma < \frac{1}{2}$ such that

$$(\nabla^j V)(1+H^2)^{-\gamma} \in \mathcal{J}$$

for each multi-index j. We then define

$$|||V|||_{\mathcal{J},H,n,\gamma} := \sup_{|j|<n} \left\|(\nabla^j V)(1+H^2)^{-\gamma}\right\|_{\mathcal{J}}.$$

Proposition 1.4.19 *Let H, V etc. be as in Definition 1.4.18. For any function $f \in \mathcal{S}^\beta(\mathbb{R})$ with $\beta > -1 + 2\gamma$ and any multi-index j (with $j = 0$ being expressly allowed) there is a constant $c_{f,j,\gamma}$ independent of H and V such that*

$$\|\nabla^j(f(H+V) - f(H))\|_{\mathcal{J}} \leq c_{f,j,\gamma} \|f\|_{\mathcal{S}_{|j|}^{-(1-2\gamma)}}$$

$$\sup_{\substack{0<k<|j| \\ m_1,\dots,m_k\geq 1 \\ \sum_i m_i = j}} \prod_{i=1}^k |||H|||_{m_i,\frac{1}{2}}^{|j|} \left(1 + \sup_{\substack{0<k<|j| \\ m_1,\dots,m_k\geq 1 \\ \sum_i m_i = j}} \prod_{i=1}^k \|\nabla^{m_i} V\|\right)^{|j|} \sup_{|k|\leq j} |||V|||_{\mathcal{J},H,k,\gamma}.$$

In particular, $f(H+V) - f(H)$ and all its derivatives lie in $\mathcal{J}$.

Proof. The proof is very similar to the one of Proposition 1.4.13, one needs to show that the smooth functional calculus for $\nabla^j(f(H+V)-f(H))$ gives an integral that is bounded absolutely in the norm of $\mathcal{J}$, such that the strict continuity implies that the integral (being a limit of Riemann sums) lies in $\mathcal{J}$. For that one expands the derivative ∇^j of

$$(H+V+z)^{-1}-(H+z)^{-1}=(H+z)^{-1}V(H+V+z)^{-1},$$

and finds it is a sums of terms of the form $a_{j,k}(z)\nabla^{j_{k+1}}(V)b_{j,k}(z)$ with

$$a_{j,k}(z)=(H+z)^{-1}\left(\prod_{m=1}^{k}\left((\nabla^{j_m}H)(H+z)^{-1}\right)\right)$$

$$b_{j,k}(z)=(H+V+z)^{-1}\prod_{m=k+2}^{n}\left((\nabla^{j_m}(H+V))(H+V+z)^{-1}\right)$$

where $\sum_{m=1}^{n} j_m=j$. We can switch the brackets in $a_{j,k}(z)$ and use that

$$\left\|(H+z)^{-1}(\nabla^k V)(H+V+z)^{-1}\right\|_{\mathcal{J}}=\left\|(H+V+\overline{z})^{-1}(\nabla^k V)(H+\overline{z})^{-1}\right\|_{\mathcal{J}}$$
$$\leq|\Im m\, z|^{-1-(1-2\gamma)}\left(1+\frac{\langle\Re e\, z\rangle}{|\Im m\, z|}\right)^{2\gamma}|||V|||_{\mathcal{J},H,k,\gamma}.$$

The remaining factors are estimated as in the proof of Lemma 1.4.13, but for simplicity with $\eta=\frac{1}{2}$,

$$\prod_{m=1}^{k}\left\|(H+z)^{-1}(\nabla^{j_m}H)\right\|\leq\left(1+\frac{\langle\Re e\, z\rangle}{|\Im m\, z|}\right)^{k}\prod_{m=1}^{k}|||H|||_{j_m,\frac{1}{2}}$$

and

$$\prod_{m=k+2}^{n}\left\|(\nabla^{j_m}(H+V))(H+V+z)^{-1}\right\|$$
$$\leq\left(1+\frac{\langle\Re e\, z\rangle}{|\Im m\, z|}\right)^{(n-k-1)\eta}\prod_{m=k+2}^{n}|||H+V|||_{j_m,\frac{1}{2}}$$

for $n \leq k+2$. According to Lemma A.4 one can therefore estimate

$$\begin{aligned}&\|\nabla^j(f(H+V)-f(H))\|_{\mathcal{J}}\\&\quad\leq \frac{1}{2\pi}\int_{\mathbb{C}}\left|(\partial_{\overline{z}}\tilde{f}_K)(z)\right|\left\|\nabla^j(H+z)^{-1}V(H+V+z)^{-1}\right\|_{\mathcal{J}}\,\mathrm{d}z\wedge\mathrm{d}\overline{z}\end{aligned}$$

in terms of the norm $\|f\|_{\mathcal{S}_K^{1-2\gamma}}$ for $K > |j|+1$. For the dependence of the constants on H one must note that, as is implicit in the proof of Proposition 1.4.16, the constants $|||H+V|||_{j_m,\frac{1}{2}}$ can be bounded in terms of $|||H|||_{j_m,\frac{1}{2}}$ and $\left\|\nabla^k V\right\|$. Taking all combinations into account one can bound the integral from the smooth functional calculus by the lengthy expression given in the statement of the Proposition. □

The norm bound is far from optimal but sufficient for our purposes; the most important thing for us is that the right-hand-side converges to 0 with the norms of $\nabla^j V$, which shows locally uniform continuity w.r.t. V under continuous perturbations.

This Proposition has many applications, e.g. if $(H+\imath)^{-1} \in M(\mathcal{A})$ and $V \in \mathcal{A}$ for some C^*-algebra $\mathcal{A}$ (or vice versa) then the perturbation of the bounded transform $F(H+V)-F(H)$ lie in $\mathcal{A}$ with all their derivatives. One of the most important applications is also that if $(H+\imath)^{-1} \in \mathcal{A} \cap L^p(\mathcal{A},\mathcal{T})$ for some non-commutative L^p-space then perturbations of the bounded transform and its derivatives have the same L^p-regularity.

1.5 K-theory and multipliers

In the standard picture the abelian group $K_0(\mathcal{A})$ of a local C^*-algebra $\mathcal{A}$ can be written as

$$K_0(\mathcal{A}) = \{[e]_0 - [s(e)]_0, \quad e \in \bigcup_{N\in\mathbb{N}} \mathcal{P}_N(\mathcal{A})\},$$

with $\mathcal{P}_N(\mathcal{A})$ the set of projections in $M_N(\mathcal{A}^\sim)$ and $s: \mathcal{A}^\sim \to \mathbb{C}$ the scalar part. Likewise,

$$K_1(\mathcal{A}) = \{[u]_1, \quad u \in \bigcup_{N\in\mathbb{N}} \mathcal{U}_N(\mathcal{A})\},$$

where $\mathcal{U}_N(\mathcal{A})$ is the set of unitaries $u \in M_N(\mathcal{A}^\sim)$ with $s(u) = \mathbb{1}_N$. In both cases the group structure on the formal differences is obtained by taking the Grothendieck group induced from the respective abelian semigroup of stable homotopy classes with the direct sum [20, 107]. If $\mathcal{A}$ is unital one can drop the unitizations and directly work with classes represented by matrices over $\mathcal{A}$ (called the unital picture of K-theory).

For convenience of the reader we recall the definition of the connecting maps:

Definition 1.5.1 *Let* $0 \to \mathcal{E} \overset{i}{\to} \hat{\mathcal{A}} \overset{q}{\to} \mathcal{A} \to 0$ *be an exact sequence of local* C^*-*algebras. Then one has the six-term exact sequence*

$$\begin{array}{ccccc} K_0(\mathcal{E}) & \xrightarrow{i_*} & K_0(\hat{\mathcal{A}}) & \xrightarrow{q_*} & K_0(\mathcal{A}) \\ \mathrm{Ind}\uparrow & & & & \downarrow\mathrm{Exp} \\ K_1(\mathcal{A}) & \xleftarrow[q_*]{} & K_1(\hat{\mathcal{A}}) & \xleftarrow[i_*]{} & K_1(\mathcal{E}) \end{array} \tag{1.5.1}$$

with the connecting maps $\mathrm{Exp} : K_0(\mathcal{A}) \to K_1(\mathcal{E})$ *and* $\mathrm{Ind} : K_1(\mathcal{A}) \to K_0(\mathcal{E})$ *defined as follows:*

For a projection $e \in M_N(\mathcal{A}^\sim)$ *defining a class* $[e]_0 - [s(e)]_0 \in K_0(\mathcal{A})$ *choose a self-adjoint contraction* $\hat{e} \in M_N(\hat{\mathcal{A}}^\sim)$ *with* $q(\hat{e}) = e$*, then*

$$\mathrm{Exp}([e]_0 - [s(e)]_0) = [e^{-\imath 2\pi\hat{e}}]_1.$$

For a unitary $u \in M_N(\mathcal{A}^\sim)$ *defining a class* $[u]_1 \in K_1(\mathcal{A})$ *choose a unitary* $\hat{w} \in M_{2N}(\hat{\mathcal{A}}^\sim)$ *such that* $q(\hat{w}) = u \oplus u^*$ *then set*

$$\mathrm{Ind}([u]_1) = [\hat{w}(\mathbb{1}_N \oplus 0_N)\hat{w}^*]_0 - [\mathbb{1}_N \oplus 0_N]_0.$$

In particular when $\mathcal{A}$ is a non-unital C^*-algebra one sometimes runs into the problem that certain constructions naturally yield elements of $K_0(\mathcal{A})$ that are difficult to express in the standard picture. For example, it is understood that any pair of projections $p, q \in M^s(\mathcal{A}) = M(\mathcal{A} \otimes \mathbb{K})$ in the stable multiplier algebra with $p - q \in \mathcal{A} \otimes \mathbb{K}$ define an element of $K_0(\mathcal{A})$. The most conceptual way to see this is in the Cuntz picture of KK-theory. Recall that [20]

$$KK(\mathcal{A}, \mathcal{B}) \simeq \{[\phi_0, \phi_1] : \phi_i : \mathcal{A} \to M^s(\mathcal{B}), \quad (\phi_0 - \phi_1)(\mathcal{A}) \subset \mathcal{B} \otimes \mathbb{K}\}$$

where the ϕ_i are $*$-homomorphisms and $[\cdot]$ denotes the homotopy class within those so-called quasihomomorphisms. For any projection e one has a homomorphism $\phi_e : \mathbb{C} \to M^s(\mathcal{A}), \lambda \mapsto \lambda e$, thus one can associate to a pair (p, q) the element

$[(\phi_p, \phi_q)] \in KK(\mathbb{C}, \mathcal{A}) \simeq K_0(\mathcal{A})$. Another (equivalent) way to construct such an element of $K_0(\mathcal{A})$ is by suspension: For $u \in C_0(\mathbb{R})^\sim$ a unitary of winding number 1 the pair (p, q) defines the unitary $[(up + 1 - p)(u^*q + 1 - q)]_1 \in K_1(S\mathcal{A})$ which is the suspension of an element in $K_0(\mathcal{A})$. Neither construction gives an immediately obvious representative in standard form $[e]_0 - [s(e)]_0$ even though it must exist. Since particular projections are preferred by the physical motivation in our applications we opt in this work to just carry along the pairs of projections that define elements of $K_0(\mathcal{A})$ without worrying how precisely it can be represented in standard form. An efficient calculus for that is given by algebras of pairs:

Definition 1.5.2 *Let $\mathcal{A}$ be a closed ideal of the C^*-algebra $\mathcal{B}$ then define the algebra of pairs*

$$\mathbb{P}(\mathcal{B}, \mathcal{A}) = \{(b_1, b_2) \in \mathcal{B} \oplus \mathcal{B} : \, b_1 - b_2 \in \mathcal{A}\}$$

with component-wise sum and product.

This is also a rather trivial example of a pullback algebra. The following is obvious but we state it explicitly for definiteness:

Lemma 1.5.3 *If $\mathcal{B}$ is unital, $K_i(\mathcal{B}) = 0$ for both $i = 0, 1$ and $\mathcal{A}$ is a closed ideal in $\mathcal{B}$ then*

$$K_i(\mathbb{P}(\mathcal{B}, \mathcal{A})) \simeq K_i(\mathcal{A})$$

with the isomorphism $(I_1)_* : K_i(\mathcal{A}) \to K_i(\mathbb{P}(\mathcal{B}, \mathcal{A}))$ induced by the inclusion $I_1 : a \in \mathcal{A} \mapsto (a, 0)$ acting via

$$[x]_i - [s(x)]_i \mapsto [(x, s(x))]_i$$

(with the image in the unital picture of K-theory).

Proof. There is an exact sequence $0 \to \mathcal{A} \to \mathbb{P}(\mathcal{B}, \mathcal{A}) \to \mathcal{B} \to 0$ induced by inclusion as the left component. Since $K_i(\mathcal{B}) = 0$ the six-term exact sequence implies that the inclusion induces an isomorphism. □

The canonical choice here is the inclusion $\mathcal{A} \otimes \mathbb{K}$ into the stable multiplier algebra $M^s(\mathcal{A}) = M(\mathcal{A} \otimes \mathbb{K})$. Since $K_i(\mathcal{A}) \simeq K_i(\mathcal{A} \otimes \mathbb{K})$ this yields a very general multiplier picture of K-theory: Any formal difference

$$[p]_0^M - [q]_0^M := (I_1)_*^{-1} \left[(p, q)\right]_0$$

with $p, q \in M^s(\mathcal{A})$ such that $p - q \in \mathcal{A} \otimes \mathbb{K}$ defines an element of $K_0(\mathcal{A})$ and

$$[u]_1^M - [v]_1^M = [uv^*]_1^M - [\mathbb{1}]_1^M = [uv^*]_1 \in K_1(\mathcal{A})$$

for unitaries $u, v \in M^s(\mathcal{A})$ such that $u - v \in \mathcal{A} \otimes \mathbb{K}$ defines an element of $K_1(\mathcal{A})$. Note that in this picture

$$[p]_0 - [s(p)]_0 = [p]_0^M - [s(p)]_0^M$$

for any element in the standard form where we naturally identify $M_N(\mathcal{A}^\sim)$ with the obvious subalgebra of $\mathcal{A}^\sim \otimes \mathbb{K}(\ell^2(\mathbb{N}))$ to handle stabilization. To simplify some notations we will also write $[p]_0^M - [q]_0^M \in K_0(\mathcal{A})$ for matrices $p, q \in M_N(\mathbb{C}) \otimes M(\mathcal{A})$ with $p - q \in M_N(\mathcal{A})$ where matrices are considered as compact operators by once and for all choosing a countable family of matrix units for $\mathbb{K}$.

For $K_1(\mathcal{A})$ there is always an obvious representative in the standard picture, but that is apparently not the case for $K_0(\mathcal{A})$.

As we will see, it is very convenient to do computations with classes in the multiplier picture, in particular, let us examine how to compute the connecting maps in K-theory without having to find a representative in standard form. Given an exact sequence

$$0 \to \mathcal{E} \overset{i}{\to} \hat{\mathcal{A}} \overset{q}{\to} \mathcal{A} \to 0$$

with $\hat{\mathcal{A}}$ and $\mathcal{A}$ *separable C^**-algebras there is an induced sequence

$$0 \to \mathrm{Ker}(\overline{q}) \to M^s(\hat{\mathcal{A}}) \overset{\overline{q}}{\to} M^s(\mathcal{A}) \to 0$$

since q extends to a map $q : \hat{\mathcal{A}} \otimes \mathbb{K} \to \mathcal{A} \otimes \mathbb{K}$ and from there to a unique surjection of the multiplier algebras $\overline{q} : M^s(\hat{\mathcal{A}}) \to M^s(\mathcal{A})$ [1, Theorem 4.2]. There is a commutative diagram with exact lines

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \hat{\mathcal{A}} & \overset{q}{\longrightarrow} & \mathcal{A} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{P}(\mathrm{Ker}(\overline{q}), \mathcal{E} \otimes \mathbb{K}) & \longrightarrow & \mathbb{P}(M^s(\hat{\mathcal{A}}), \hat{\mathcal{A}} \otimes \mathbb{K}) & \overset{\overline{q} \oplus \overline{q}}{\longrightarrow} & \mathbb{P}(M^s(\mathcal{A}), \mathcal{A} \otimes \mathbb{K}) & \longrightarrow & 0 \end{array} \tag{1.5.2}$$

where the vertical arrows are of the form $x \mapsto (x \otimes e, 0)$ with e an arbitrary rank one projection. Since $K_i(M^s(\hat{\mathcal{A}})) = 0 = K_i(M^s(\mathcal{A}))$ the six-term sequence implies $K_i(\mathrm{Ker}(\overline{q}) = 0$ and hence all vertical arrows induce isomorphisms of the K-groups by Lemma 1.5.3. The naturalness of the connecting maps therefore implies that one can equivalently use the connecting map of either the top or bottom sequence. By definition of the multiplier picture we obtain immediately

Proposition 1.5.4 *Let $0 \to \mathcal{E} \overset{i}{\to} \hat{\mathcal{A}} \overset{q}{\to} \mathcal{A} \to 0$ be an exact sequence of separable C^*-algebras.*

For a pair of projections $(p, q) \in \mathbb{P}(M^s(\mathcal{A}), \mathcal{A} \otimes \mathbb{K})$ defining a class $[e_+]_0^M - [e_-]_0^M \in K_0(\mathcal{A})$ choose a self-adjoint contraction $(\hat{e}_+, \hat{e}_-) \in \mathbb{P}(M^s(\hat{\mathcal{A}}), \hat{\mathcal{A}} \otimes \mathbb{K})$ with $(\overline{q}(\hat{e}_+), \overline{q}(\hat{e}_-)) = (e_+, e_-)$, then

$$\mathrm{Exp}([e_+]_0^M - [e_-]_0^M) = [e^{-\imath 2\pi \hat{e}_+}]_1^M - [e^{-\imath 2\pi \hat{e}_-}]_1^M.$$

For a pair of unitaries $(u_+, u_-) \in \mathbb{P}(M^s(\mathcal{A}), \mathcal{A} \otimes \mathbb{K})$ defining a class $[u_+]_1^M - [u_-]_1^M \in K_1(\mathcal{A})$ choose a unitary lift $\hat{w} = (\hat{w}_+, \hat{w}_-) \in M_2(\mathbb{C}) \otimes \mathbb{P}(M^s(\hat{\mathcal{A}}), \hat{\mathcal{A}} \otimes \mathbb{K})$ of $(u_+ \oplus u_+^*, u_- \oplus u_-^*)$ then

$$\mathrm{Ind}([u_+]_1^M - [u_-]_1^M) = [\hat{w}_+(\mathbb{1}_N \oplus 0_N)\hat{w}_+^*]_0^M - [\hat{w}_-(\mathbb{1}_N \oplus 0_N)\hat{w}_-^*]_0^M.$$

This formulation already incorporates some of the simplifications from the unital picture of K-theory and it is without loss of generality unnecessary to involve larger matrices due to stability.

1.6 Cyclic cocycles and pairings

An important way to assign numerical invariants to K-theory classes is via the pairings with cyclic cohomology [39]. We only recall as many details as are absolutely necessary, since our applications of cyclic cohomology are very basic.

Definition 1.6.1 *A cyclic n-cocycle on an algebra $\mathscr{A}$ is an $n+1$-linear functional $\varphi : \mathscr{A}^{n+1} \to \mathbb{C}$ which is cyclic*

$$\varphi(a_0, \dots, a_n) \;=\; (-1)^n \varphi(a_1, \dots, a_n, a_0)$$

and a cocycle w.r.t. the Hochschild boundary operator b defined by

$$\begin{aligned}(b\varphi)(a_0, \dots, a_{n+1}) \;=\; & \sum_{j=0}^{n} (-1)^j \varphi(a_0, \dots, a_j a_{j+1}, \dots, a_{n+1}) \\ & + (-1)^{n+1} \varphi(a_{n+1} a_0, a_1, \dots, a_n)\;,\end{aligned}$$

that is, $b\varphi = 0$.

Any cyclic cocycle naturally extends to a cyclic cocycle on $M_N(\mathscr{A}^\sim)$ via an equivalent formulation in terms of n-cycles. Assume that $\mathscr{A}$ is an m-convex Fréchet algebra (thus closed under holomorphic functional calculus) and that φ is continuous. We say that $\mathscr{A}$ is smooth in a C^*-algebra $\mathcal{A}$ if $M_N(\mathscr{A})$ is densely and continuously included in $M_N(\mathcal{A})$ for each N and the inclusion is spectrally invariant, i.e. $\sigma_A(a) = \sigma_{\mathcal{A}}(a)$ for all $a \in M_N(\mathscr{A})$. If $\mathscr{A}$ is smooth in $\mathcal{A}$ then any even cyclic cocycle on $\mathscr{A}$ defines a pairing with $K_0(\mathcal{A})$ by setting

$$\langle [\varphi], [e]_0 - [\mathbb{1}_K]_0 \rangle \;=\; \varphi(e, e, \dots, e)$$

for any representative $e \in M_N(\mathscr{A}^\sim)$ and any odd cyclic cocycle defines a pairing with $K_1(\mathcal{A})$ by setting

$$\langle [\varphi], [u]_1 \rangle \;=\; \varphi(u^{-1} - 1, u - 1, u^{-1} - 1, \dots, u - 1)$$

for $[u]_1$ represented by any invertible $u \in M_N(\mathscr{A}^\sim)$ [38].

To prove the spectral invariance one can often apply a useful sufficient criterion [114]:

Theorem 1.6.2 *Let $\mathscr{A}$ be a Fréchet algebra with jointly continuous multiplication and with topology generated by the increasing family of seminorms $(\|\cdot\|_j)_{j\in\mathbb{N}}$.*

Assume that $\mathscr{A}$ is densely and continuously included into the C^*-algebra $\mathcal{A}$, in such a way that $\|\cdot\|_0$ is the C^*-norm.

We say that $\mathscr{A}$ is strongly spectral invariant in $\mathcal{A}$ if there is a constant $C > 0$ such that for every $j \in \mathbb{N}$ there is some $D_j > 0$ and $p_j \in \mathbb{N}$ such that for all $a_1, \dots, a_n \in \mathscr{A}$, one has

$$\|a_1 \cdots a_n\|_j \le D_j\, C^{\,n} \sum_{j_1+\dots+j_n \le p_j} \|a_1\|_{j_1} \cdots \|a_n\|_{j_n} \;, \tag{1.6.1}$$

independently of $n \in \mathbb{N}$.

If $\mathscr{A}$ is strongly spectral invariant in $\mathcal{A}$ then $M_N(\mathscr{A})$ is spectral invariant in $M_N(\mathcal{A})$ for each $N \in \mathbb{N}$.

The important point of (1.6.1) is that in the product on the right-hand side at most p_j of the $j_1, \dots, j_n$ are larger than zero, thus the norms of a power $\|x^n\|_j$ grow with n as $C^n \|x\|^{n-p_j}$ which can be used to prove convergence of Neumann series. While the condition looks complicated at first, it is often straightforward to check in practice. For example, for a C^*-algebra $\mathcal{A}$ supplied with a strongly continuous $\mathbb{R}^d$-action θ the algebra of smooth elements is strongly spectral invariant in $\mathcal{A}$ due to

$$\left\|\nabla^j(a_1 ... a_n)\right\| \leq \sum_{j_1+...+j_n=j} \left\|\nabla^{j_1} a_1\right\| ... \left\|\nabla^{j_n} a_n\right\|.$$

An important construction is the suspension of cocycles which is used to shift the degrees of the pairings (and which must be contrasted with suspension map of [39] which would be the double suspension in the terminology here):

Theorem 1.6.3 *Let $\mathscr{A}$, $\mathcal{A}$ be a Fréchet algebra and C^*-algebra as in Theorem 1.6.2. Denote by $\mathcal{S}(\mathbb{R}, \mathscr{A})$ the $\mathscr{A}$-valued Schwartz functions with their natural Fréchet topology then the inclusion $\mathcal{S}(\mathbb{R}, \mathscr{A}) \to S\mathcal{A}$ is strongly spectral invariant.*

If $\varphi : \mathscr{A}^{n+1} \to \mathbb{C}$ is a continuous cyclic cocycle then its suspension

$$\varphi^s : \mathcal{S}(\mathbb{R}, \mathscr{A})^{n+1} \to \mathbb{C}$$

given by

$$\begin{aligned}
&\varphi^s(f_0, ..., f_{n+1}) \\
&= \sum_{j=1}^{n+1} (-1)^{n+2-j} \int_{\mathbb{R}} \varphi(f_0(t), ..., f_{j-2}(t), f_{j-1}(t)\dot{f}_j(t), f_{j+1}(t), ..., f_{n+2}(t)) \frac{\mathrm{d}t}{2\pi}
\end{aligned}$$

is again a continuous cyclic cocycle and the pairings with $K_i(S\mathcal{A})$ are dual to the suspension maps $\Psi_i : K_i(\mathcal{A}) \to K_{i+1}(S\mathcal{A})$ in the sense that

$$\langle [\varphi^s], \Psi_i(x) \rangle = (-1)^{j+1} \frac{c_n}{c_{n+1}} \langle [\varphi], x \rangle$$

with the constants

$$c_n = \begin{cases} \frac{(2\pi\imath)^k}{k!}, & \text{for } n = 2k\,, \\ \frac{\imath\,(\pi\imath)^k}{(2k+1)!!}, & \text{for } n = 2k+1\,. \end{cases}$$

The theorem is well-known in various forms (see [97, 74, 75, 111] among others) and the formulation here is drawn from [111, Section 4.4]. One can also absorb the normalization constants into the definition of the pairings (as it is done in [39]) but we do not adopt that convention in this work.

2 Index theorems for n-parameter actions

In this chapter we consider a W^*-dynamical system $(\mathcal{M}, G, \alpha)$ with a von Neumann algebra $\mathcal{M}$ equipped with an α-invariant n.s.f. trace $\mathcal{T}$ and where G is an n-parameter group $G = \mathbb{T}^{n_0} \times \mathbb{R}^{n_1}$. From this data one has natural pairings between regular enough projections or unitaries in $\mathcal{M}$ with cyclic cocycles, which give geometric invariants that we call Chern numbers. In the simplest case of a one-parameter action the invariant in question is the non-commutative winding number. One can establish for these Chern number semifinite index theorems over the von Neumann algebra $\mathcal{M} \rtimes_\alpha G$, i.e. write them as indices of certain operators in $\mathcal{M} \rtimes_\alpha G$ constructed from a symbols in $\mathcal{M}$. This constitutes a non-commutative analogue of the Gohberg-Krein theorem. Those considerations can be made in a smooth C^*-algebraic setting (see e.g. [81, 96] for the one-dimensional case, [10] for the higher-dimensional) but also for symbols which have as little regularity as possible. The classical example is the Toeplitz index theorem

$$\mathrm{Ind}(P\pi(u)P^*) = \mathrm{Wind}(u)$$

for the partial isometry $P : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{Z})$ and $\pi(u)$ the Laurent-operator associated to a unitary symbol $u \in L^\infty(\mathbb{T})$. The equality holds true for a smooth symbol, but also for a certain quasi-continuous ones, for example the Besov class $B^{1/p}_{p,p}(\mathbb{T})$ for $1 < p \leq 2$ is sufficient to make sense of the winding number and of the index theorem [39]. An index theorems for such non-smooth elements was first proved in [16], generalized in [100, 102, 103] to higher and odd-dimensional cases, and, building on that, extended to semi-finite index theorems with fractional smoothness conditions in [111, Chapter 3]. The motivation was originally to give meaning to topological invariants in mobility-gapped topological insulators (which we will study in Chapter 4). The regularity that is present in such a case has no immediate classical analogue since it requires Sobolev-embedding to fail. More precise sufficient conditions for the index theorems were derived in [111] and will be explained in more detail in the following.

2.1 Crossed products with n-parameter groups

In this section G will always denote an abelian n-parameter group of the form $G = \mathbb{T}^{n_0} \oplus \mathbb{R}^{n_1}$ where $n = n_0 + n_1$ and the torus is parametrized as $\mathbb{T} = \mathbb{R}/\mathbb{Z} \cong [0, 1)$.

Consider a C^*-dynamical system $(\mathcal{A}, \alpha, G)$ with $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}_0$. One can then form the regular representation $\pi \times U$ acting on $L^2(G, \mathcal{H}_0)$. The generators $U(t)$ of G are then explicitly given by exponentiation of the commuting self-adjoint generators

$$D_j = \imath \partial_j, \qquad U(t) = e^{2\pi\imath D\cdot t}. \tag{2.1.1}$$

Compactly supported continuous functions $f \in C_c(\hat{G})$ can be written in terms of the Fourier transform

$$(\mathcal{F}f)(k) \;=\; \int_G \overline{\langle k, t\rangle}\, f(t)\, \mathrm{d}t\,, \qquad f \in L^1(G)$$

with the character $\langle k, t\rangle = e^{-2\pi\imath k\cdot t}$ and inverse

$$(\mathcal{F}^{-1}g)(t) \;=\; \int_{\hat{G}} \langle k, t\rangle\, g(k)\, \mathrm{d}k\,, \qquad g \in L^1(\hat{G}).$$

Therefore $g(D) \;=\; \big(\mathcal{F}(\mathcal{F}^{-1}g)\big)(D) \;=\; \int_G (\mathcal{F}^{-1}g)(t)\, e^{2\pi\imath D\cdot t}\, \mathrm{d}t$ defines a multiplier of $\mathcal{A} \rtimes_\alpha G$. Furthermore $\pi(a)g(D)$ is an element of the C^*-crossed product and indeed the linear span of such elements is dense, i.e. any $\hat{a} \in \mathcal{A} \rtimes_\alpha G$ can be approximated in operator norm by

$$\hat{a} = \sum_{n=1}^{N} \pi(a_n) g_n(D)$$

up to an arbitrarily small error. To see this one recalls that elements of the form $\int_G f(t)U(t)\mathrm{d}t$ with $f \in C_c(G, \mathcal{A})$ are norm-dense and that one can approximate $f = \sum_{n=1}^{N} a_n \otimes f_n$.

Let us now consider the case of a W^*-dynamical system $(\mathcal{M}, G, \alpha)$. Clearly, $\mathcal{M} \rtimes_\alpha G$ is generated by $\pi(\mathcal{M})$ and the set of all bounded Borel functions $f(D)$ of the generators. In particular $D_1, \ldots, D_n$ are unbounded operators that are affiliated to the crossed product $\mathcal{M} \rtimes_\alpha G$.

It is important to clarify the relation of the abstract generators D with the generators of a covariant representation. Often we already have a covariant representation (ρ, V) of our dynamical system where V is implemented by exponentiation of some commuting unbounded operators on a Hilbert space $\mathcal{H}_0$. To apply the crossed product to the study of such a representation one can transform the regular representation:

Proposition 2.1.1 *Let $\mathcal{M}$ act on a Hilbert space $\mathcal{H}_0$ with n commuting unbounded operators $X_1, ..., X_d$ such that*

$$\alpha_t(m) = e^{2\pi\iota X\cdot t} m e^{-2\pi\iota X\cdot t}.$$

Then the defining representation of $\mathcal{M} \rtimes_\alpha G$, i.e. one defined from the regular representation on $L^2(G, \mathcal{H}_0)$ is unitarily equivalent to the W^*-algebraic span of the operators

$$\mathbb{1}_{L^2(\hat{G})} \otimes \mathcal{M}, \qquad f(\hat{X}) = \int_{\hat{G}}^{\oplus} f(X-k)\,\mathrm{d}k$$

on $L^2(\hat{G}, \mathcal{H}_0)$ where integration is w.r.t. a Haar measure on $\hat{G}$ and one uses the span of all bounded Borel functions f on $\hat{G}$.

Proof. We begin with the regular representation on the Hilbert space $L^2(G, \mathcal{H})$, given by the generators

$$(\pi^{\text{reg}}(a)\psi)(t) = \alpha_{-t}(a)\psi(t),$$
$$(U_s^{\text{reg}}\psi)(t) = \psi(t-s)$$

for $\psi \in L^2(\mathbb{R}, \mathcal{H})$, $a \in \mathcal{M}$, $t, s \in G$. Then $\pi^{\text{reg}} \times U^{\text{reg}}$ is a faithful representation since it is constructed from a faithful regular representation. It is a well-known trick that a regular representation can be written in terms of an existing covariant representation $(\pi, U) = (\text{id}, e^{2\pi\iota X\cdot})$ by applying the involutive unitary

$$(W\psi)(t) = U(-t)\psi(-t)$$

which transforms the generators to

$$(W\pi^{\text{reg}}(a)W\psi)(t) = a\,\psi(t),$$
$$(WU_s^{\text{reg}}W\psi)(t) = U(s)\psi(s+t).$$

After Fourier transform in the first component this becomes

$$(\mathcal{F}W\pi^{\text{reg}}(a)W\mathcal{F}^*\hat{\psi})(\gamma) = a\hat{\psi}(k),$$
$$(\mathcal{F}WU_s^{\text{reg}}W\mathcal{F}^*\hat{\psi})(\gamma) = \langle k, s\rangle U(s)\hat{\psi}(k).$$

for $\hat{\psi} \in L^2(\hat{G}, \mathcal{H})$ and $\gamma \in \hat{G}$. Since

$$\overline{\langle k, s\rangle} U(s) = e^{2\pi\iota(X-k)\cdot s}$$

one has

$$\widehat{U}_s := \mathcal{F}WU_s^{\mathrm{reg}}W\mathcal{F}^* = \int_{\hat{G}}^{\oplus} e^{2\pi\iota(X-k)\cdot s}\mathrm{d}k$$

and the same direct sum decomposition applies to any Borel function of D. □

Thus the elements of a crossed product are direct integrals of operators with fibers in the initial spatial representation. In a regular representation (using notation as in the proof above) the dual action is implemented via conjugation with the unitary

$$(V(k)\psi)(t) = \langle k,t\rangle\psi(t)$$

under our identification $\hat{G} = \mathbb{Z}^{n_0} \oplus \mathbb{R}^{n_1}$. In particular one has the relations

$$V(k)\pi^{\mathrm{reg}}V(k)^* = \pi^{\mathrm{reg}}, \quad V(k)U_s^{\mathrm{reg}}V(k)^* = \overline{\langle k,s\rangle}U_s^{\mathrm{reg}}$$

the second of which is called the Heisenberg commutation relation. In the spatial picture, the dual action therefore acts by shifting the generators X to $X-k$. Sometimes it happens that the representation on $\mathcal{H}_0$ already extends to a faithful covariant representations of the full crossed product without having to enlarge the Hilbert space. From the above one can see that this is the case when X and $X-k$ are unitarily equivalent for all $k \in \hat{G}$:

Proposition 2.1.2 *A generalized regular covariant representation of a C^*- (or W^*-)dynamical system $(\mathcal{B}, G, \alpha)$ shall be a couple (π, U, V) with (π, U) a (normal) covariant representation and $V : \hat{G} \to \mathcal{U}(\mathcal{H})$ a strongly continuous representation such that*

$$V(k)\pi(a)V(k)^* = \pi(a)\,, \qquad \forall a \in \mathcal{B}, k \in \hat{G}$$

and

$$V(k)U(t)V(k)^* = \overline{\langle k,t\rangle}U(t), \qquad \forall k \in \hat{G}, t \in G.$$

Assume that π is faithful then in the C^*-dynamical case the integrated form $(\pi \times U)$ yields a faithful representation of $\mathcal{B} \rtimes_\alpha G$ and in the W^*-dynamical case there is an isomorphism from $\mathcal{B} \rtimes_\alpha G$ to the W^*-algebraic span of $\pi(\mathcal{M})$ and U.

Proof. Identifying (π, U) with $(\mathrm{id}, e^{\iota 2\pi X\cdot})$ we are in the situation of Proposition 2.1.1. By conjugating in the representation on $L^2(\hat{G}, \mathcal{H}_0)$ of Proposition 2.1.1 with the unitary

$$(W_2\psi)(k) = V(k)\psi(k)$$

the generators are mapped to

$$W_2(\mathbb{1}_{L^2(\hat{G})} \otimes a)W_2^* = \mathbb{1}_{L^2(\hat{G})} \otimes a\,, \quad W_2\hat{U}(t)W_2^* = \mathbb{1}_{L^2(\hat{G})} \otimes U(t).$$

Thus dropping the trivial tensor factor gives the desired isomorphism. □

In other words, if the dual action can be implemented via a spatial family of unitaries that commutes with $\mathcal{M}$ then the span of the generators is equivalent to the crossed product. This is related to an alternative description of crossed products with abelian groups as so-called G-products (see e.g. [92]).

One can also compute the dual trace for generators of the crossed product. Let therefore $\mathcal{T}$ be an α-invariant n.s.f. trace on $\mathcal{M}$ and let $\hat{\mathcal{T}}_\alpha$ be the dual trace defined using the Haar measure on G which assigns measure 1 to the unit cube, i.e. $\mu(\mathbb{T}^n) = 1$ in our presentation.

Proposition 2.1.3 *For $f, g \in L^2(\hat{G}) \cap L^\infty(\hat{G})$ and $a, b \in L^2(\mathcal{M}, \mathcal{T}) \cap \mathcal{M}$ one has*

$$\hat{\mathcal{T}}_\alpha(\pi(a)^*\overline{f}(D)g(D)\pi(b)) = \langle f, g\rangle_{L^2(\hat{G})}\,\mathcal{T}(a^*b) = \hat{\mathcal{T}}_\alpha(\overline{f}(D)\pi(a)^*\pi(b)g(D))$$

and in particular those elements are $\hat{\mathcal{T}}_\alpha$ trace-class.

The proof is obvious from the definition of the dual trace and the Plancherel theorem. Since any $a \in L^1(\mathcal{M})$ can be factored as a product, a simple corollary is

Corollary 2.1.4 *For $f, g \in L^2(\hat{G}) \cap L^\infty(G)$ and $a \in L^1(\mathcal{M}, \mathcal{T}) \cap \mathcal{M}$ one has*

$$\hat{\mathcal{T}}_\alpha(\overline{f}(D)\pi(a)g(D)) = \langle f, g\rangle_{L^2(\hat{G})}\,\mathcal{T}(a).$$

The order is important here since $\pi(a)g(D)$ can fail to be trace-class. For such generators the L^p-properties follow closely the Birman-Solomyak theory for the Schatten-class properties of so-called $f(X)g(\nabla)$-operators (see [115, Chapter 4]):

Proposition 2.1.5 ([111, Proposition 1.5.5]) *Let $2 \leq p \leq \infty$ and $f \in L^p(\hat{G})$. The map*

$$(a, f) \in (\mathcal{M} \cap L^p(\mathcal{M})) \times (L^\infty(\hat{G}) \cap L^p(\hat{G})) \mapsto \pi(a)f(D) \in \mathcal{M} \rtimes_\alpha G$$

is $L^p(\mathcal{M}) \times L^p(\hat{G}) \to L^p(\mathcal{M} \rtimes_\alpha G)$-bounded with

$$\left\|\pi(a)f(D)\right\|_p \;\leq\; \|a\|_{L^p(\mathcal{M})}\,\|f\|_{L^p(\hat{G})}\;. \tag{2.1.2}$$

For $p = 2$, one even has equality

$$\left\|\pi(a)f(D)\right\|_2 \;=\; \|a\|_{L^2(\mathcal{M})}\,\|f\|_{L^2(\hat{G})}\,. \tag{2.1.3}$$

The proof is simple interpolation between the endpoint $p = 2$ that we saw above and $p = \infty$ which is trivial. Since it is very important for us let us also include the two-sided version in detail:

Proposition 2.1.6 *Let $1 \leq p \leq \infty$ and $f, g \in L^{2p}(\hat{G})$. The map*

$$(a, f, g) \in (\mathcal{M} \cap L^p(\mathcal{M})) \times (L^\infty(\hat{G}) \cap L^{2p}(\hat{G}))^2 \mapsto f(D)\pi(a)g(D) \in \mathcal{M} \rtimes_\alpha G$$

is $L^p(\mathcal{M}) \times L^{2p}(\hat{G})^2 \to L^p(\mathcal{M} \rtimes_\alpha G)$-bounded with

$$\left\|f(D)\pi(a)g(D)\right\|_p \;\leq\; \|a\|_{L^p(\mathcal{M})}\,\|f\|_{L^{2p}(\hat{G})}\;\|g\|_{L^{2p}(\hat{G})}\;. \tag{2.1.4}$$

Proof. Write $a = (u\,|a|^{\frac{1}{2}})\,|a|^{\frac{1}{2}}$ as its polar decomposition then

$$\begin{aligned}\left\|f(D)\pi(a)g(D)\right\|_p &\leq \left\|f(D)\pi(|a|^{\frac{1}{2}})\right\|_{2p}\left\|\pi(|a|^{\frac{1}{2}})g(D)\right\|_{2p}\\ &\leq \left\||a|^{\frac{1}{2}}\right\|^2_{L^{2p}(\mathcal{M})}\,\|f\|_{L^{2p}(\hat{G})}\;\|g\|_{L^{2p}(\hat{G})}\end{aligned}$$

and by definition $\left\||a|^{\frac{1}{2}}\right\|^2_{L^{2p}(\mathcal{M})} = \|a\|_{L^p(\mathcal{M})}$. □

The bounded extensions of the above maps make it possible to embed certain products of unbounded $\mathcal{T}$-measurable operators into the $\hat{\mathcal{T}}_\alpha$-measurable operators.

Another important use of the crossed products is to express the smoothness of elements of $\mathcal{M}$ in terms of what we call matrix elements. In the following we use spectral projections of the commuting generators the notation

$$P_I \;=\; \chi\big((D_1, \dots, D_n) \;\in\; I\big) \tag{2.1.5}$$

where $I \subset \mathbb{R}^d$ is a Borel set. The basic observation is that elements $x \in \mathcal{M}$ with compact Arveson spectrum $\sigma_\alpha(x)$ have finite hopping range between spectral subspaces:

Lemma 2.1.7 ([111, Lemma 3.1.2]) *Let $a \in \mathcal{M} \cap L^1(\mathcal{M})$ and $I, J \subset \mathbb{R}^n$ be closed sets. Then*

$$P_I\,\pi(a) \;=\; P_I\,\pi(a)\,P_{I-\sigma_\alpha(a)}\,, \qquad \pi(a)\,P_J \;=\; P_{J+\sigma_\alpha(a)}\,\pi(a)\,P_J\,. \tag{2.1.6}$$

As an exemplary application one can use this to derive L^1-norm estimates for products of the generators $\pi(a)f(D)$ which incorporate a limited amount of smoothness:

Proposition 2.1.8 ([111, Proposition 3.1.3]) *Let $Q_y = y + [-\frac{1}{2}, \frac{1}{2})^n$ be the unit cube with center y. Then $1 = \sum_{y \in \mathbb{Z}^n} \chi_{Q_y}(\lambda)$ for all λ. Now $\ell^1(L^2(\hat{G}))$ is defined as the closed subspace of $L^2(\hat{G})$ with*

$$\|f\|_{\ell^1(L^2)} \;=\; \Big(\sum_{y \in \mathbb{Z}^n} \left\|\chi_{Q_y} f\right\|_2^1 \Big) \;<\; \infty\,.$$

For $a \in \mathcal{M} \cap B^{\frac{n}{2}}_{1,1}(\mathcal{M})$ and $f \in L^\infty(\hat{G}) \cap \ell^1(L^2(\hat{G}))$, there is a constant $C > 0$ independent of a and f such that

$$\left\|\pi(a)\,f(D)\right\|_1 \;\leq\; C\,\|a\|_{B^{\frac{1}{2}}_{1,1}}\;\|f\|_{\ell^1(L^2)}\;. \tag{2.1.7}$$

Proof. Write $a = \sum_{j=0} \widehat{W_j} * a$ and $f_y = \chi_y f$. Then

$$\pi(a) f_y(D) = \sum_{j=0} \pi(\widehat{W_j} * a) f_y(D) = \sum_{j=0}^{\infty} \chi_{Q_y + \mathrm{supp}(W_j)} \pi(\widehat{W_j} * a) f_y(D)$$

and from Proposition 2.1.8 one has

$$\begin{aligned} \pi(\widehat{W_j} * a)\,f_y(D) \;&\leq\; \left\|\chi_{Q_y+\mathrm{supp}(W_j)}\right\|_{L^2(\hat{G})}\;\left\|\widehat{W_j} * a\right\|_1\;\left\|f_y\right\|_{L^2(\hat{G})} \\ &\leq\; 2^{\frac{n}{2}(j+1)}\left\|\widehat{W_j} * a\right\|_1\;\left\|f_y\right\|_{L^2(\hat{G})} \end{aligned}$$

such that one obtains the norm of $B^{\frac{n}{2}}_{1,1}$ respectively $\ell^1(L^2)$ by summing over j and y. □

2.2 Hankel operators and quantum differentiation

The n generators of the crossed product combine naturally into a self-adjoint Dirac operator

$$\mathbf{D} = \sum_{i=1}^{n} \gamma_i \otimes D_i = \gamma \cdot D , \tag{2.2.1}$$

where $\gamma_1, \dots, \gamma_n$ are generators of an irreducible representation of the complex Clifford algebra Cl_n, i.e. they are matrices in $M_N(\mathbb{C})$ with $N = \lfloor n/2 \rfloor$ which satisfy the commutation relations

$$\gamma_i^* = \gamma_i , \qquad \gamma_j \gamma_i = \gamma_i \gamma_j \text{ for } i \neq j . \tag{2.2.2}$$

Following the convention of [111] we assume that $\gamma_1 \cdots \gamma_n = \imath^{\frac{n-1}{2}} \mathbb{1}$ for odd n since there are two inequivalent representations and the right-hand side is otherwise only fixed up to a sign which would reappear in some of the index formulas. The Dirac operator is affiliated to $\mathcal{N} := M_N(\mathcal{M} \rtimes_\alpha G)$ which means that in particular its phase $\mathbf{F} = \operatorname{sgn}(\mathbf{D})$ and positive spectral projection $\mathbf{P} = \chi_{(0,\infty]}$ lie in $\mathcal{N}$. By construction of the crossed product there is a canonical inclusion $\pi : \mathcal{M} \to \mathcal{M} \rtimes_\alpha G$.

Definition 2.2.1 *Given an element $a \in \mathcal{M}$ one defines the (one-sided) Hankel operator*

$$H_a = \mathbf{P}\,\pi(a)(\mathbb{1} - \mathbf{P}) \in \mathcal{N} ,$$

the two-sided Hankel operator

$$\hat{H}_a = [\operatorname{sgn}(\mathbf{D}), \pi(a)] \in \mathcal{N} ,$$

and the Toeplitz operator are

$$T_a = \mathbf{P}\,\pi(a)\mathbf{P} \in \mathcal{N}$$

where we identify $\pi(a) \in \mathcal{M} \rtimes_\alpha G$ with $\mathbb{1}_N \otimes \pi(a) \in \mathcal{N}$, i.e. $\pi(a)$ in particular commutes with $\gamma_1, \dots, \gamma_n$. We call a the symbol of the respective operator.

For odd n we are interested in the computation of the indices of above Toeplitz operators. Let us recall the necessary definitions briefly. To any semifinite von Neumann algebra $\mathcal{N}$ with trace τ one associates the ideal of τ-compact operators $\mathcal{K}(\mathcal{N}, \tau)$. It is defined as the operator-norm-completion of $L^1(\mathcal{N}) \cap \mathcal{N}$, thus it is a C^*-algebra. A bounded operator $T \in \mathcal{N}$ whose image in the Calkin algebra $\mathcal{N}/\mathcal{K}(\mathcal{N},\tau)$ is invertible is called Breuer-Fredholm (after [34]) or τ-Fredholm and it has a real-valued index [96]

$$\tau\text{-Ind}(T) = \tau(\text{Ker}(a)) - \tau(\text{Ker}(a^*)) \in \mathbb{R}$$

with the kernel projections of the respective operators. The index is finite, since any compact projection is automatically trace-class. Like the usual Fredholm index the τ-index is invariant under norm-continuous homotopy, τ-compact perturbations and unitary equivalence.

For a unitary symbol $u \in \mathcal{M}$ the Toeplitz operator T_u is $\hat{\mathcal{T}}_\alpha$-Fredholm on the hereditary subalgebra $\mathbf{P}\mathcal{N}\mathbf{P}$ in particular if the associated Hankel operator $\hat{H}_u$ is τ-compact. The most practical criterion to decide that is to impose conditions for which the Hankel-operator is more strongly in some space $L^p(\mathcal{N})$. Moreover, large enough L^p-regularity enables computation of the index via the Fedosov-Calderon formula (for a proof that generalizes to the semifinite case see [102]):

Theorem 2.2.2 *Let $a \in \mathcal{N}$ such that $(\mathbb{1} - aa^*)^n \in L^1(\mathcal{N},\tau)$ and $(\mathbb{1} - a^*a)^n \in L^1(\mathcal{N},\tau)$ for some $n \in \mathbb{N} \setminus \{0\}$. Then a is τ-Fredholm and for all $m \geq n$,*

$$\tau\text{-Ind}(a) = \tau((\mathbb{1} - a^*a)^m) - \tau((\mathbb{1} - aa^*)^m) . \qquad (2.2.3)$$

Thus arises the problem of finding criteria which determine whether a Hankel operator is an element of some L^p-space in terms of its symbol. A special case are the classical Hankel operators which are recovered by using $\mathcal{M} = L^\infty(\mathbb{T})$ represented as Laurent operators on $\ell^2(\mathbb{Z})$ with $\mathbf{D}$ the position operator $\mathbf{D}|n\rangle = n|n\rangle$. The question of Schatten-class membership of Hankel operators was conclusively answered by Peller [95] (also in the vector-valued case [94]), namely there is a one-to-one correspondence with the Besov-regularity of its symbol. While there is no unique higher-dimensional generalization of those classical Hankel-operators for symbols in $L^\infty(\mathbb{R}^d)$, one can consider more broadly so-called paracommutators which associate to a symbol an integral operator with a certain kernel [69]. One then has a very similar relation between Schatten-classes and Besov regularity of the symbol depending on the asymptotics of the integral kernels. Those ideas were picked up in [111] to characterize the L^p-regularity of Hankel-operators with non-commutative symbols:

Theorem 2.2.3 *[111, Theorems 3.1.5,3.2.1,3.3.1] For $n = 1$ and $a \in \mathcal{M} \cap B^{1/p}_{p,p}(\mathcal{M})$ the Hankel operator satisfies $H_a, \hat{H}_a \in L^p(\mathcal{N})$ for all $1 \leq p < \infty$.*

For $n > 0$ and $a \in \mathcal{M} \cap B^{n/p}_{p,p}(\mathcal{M})$ the Hankel operator satisfies $H_a, \hat{H}_a \in L^p(\mathcal{N})$ for all $n < p < \infty$.

For the same combinations of n and p one has conversely, that the norm

$$\|a\|_p + \|H_a\|_{L^p(\mathcal{N})} + \|H_{a^*}\|_{L^p(\mathcal{N})}$$

is an equivalent norm for $B^{n/p}_{p,p}(\mathcal{M})$, i.e. if the Hankel operators H_a and H_{a^*} lie in $L^p(\mathcal{N})$ for $a \in L^p(\mathcal{M}) \cap \mathcal{M}$ then a is in the corresponding Besov space.

Instead of Besov-regularity one often prefers Sobolev-regularity since the Sobolev norms can be estimated simply by computing derivatives. While one can use the various relations between Sobolev and Besov classes there is also a more direct relation to the commutator with a Dirac-operator. Instead of classial techniques it is based on estimates for double-operator integrals and was first proved in [83], which as noted in [111] is a special case of the setup here.

Theorem 2.2.4 *[111, Theorem 3.4.1] Assume that $2 \leq n < \infty$. There is a bounded map from $W^1_n(\mathcal{M}, \alpha) \to L^{(n,\infty)}(\mathcal{N})$ extending the map $a \in W^1_n(\mathcal{M}, \alpha) \cap \mathcal{M} \mapsto [\mathrm{sgn}(\mathbf{D}), \pi(a)] \in \mathcal{N}$.*

The spaces $L^{(n,\infty)}(\mathcal{N})$ here are so-called weak L^p-spaces which are characterized by the asymptotics of their singular value function. They occur in the real interpolation scales between the usual non-commutative L^p-spaces, in particular by a generalized Marcinkiewicz interpolation theorem [48] one has $L^p(\mathcal{N}) \subset \mathcal{N} \cap L^{(n,\infty)}(\mathcal{N})$ for all $p \in (n, \infty]$. Thus there are nonstandard embeddings:

Corollary 2.2.5 *There is a continuous embedding $\mathcal{M} \cap W^1_p(\mathcal{M}) \to \mathcal{M} \cap B^{\frac{n}{n+1}}(\mathcal{M})$ for each $p \in [n, n+1]$ if $n > 1$ and $p \in (1, 2]$ for $p = 1$.*

Here one also applied the inclusions of Proposition 1.4.6. Since our symbols in index theory are projections or unitaries those inclusions always apply and thus there is rarely a need to work with Besov norms directly.

2.3 Index theorem for Chern numbers

For the purposes of index theory one generally requires matrix-valued symbols which have $(n+1)$-summable commutators with the Dirac phase $\mathbf{F}$. Furthermore, it will be useful to also have $(n+1)$-summable commutators with the phase of the shifted Dirac operator

$$\mathbf{F}_{x_0} = \operatorname{sgn}(\gamma \cdot (D + x_0))$$

for all $x_0 \in \mathbb{R}^n$ such that one can average over the offset x_0. The natural algebra of symbols which has those properties are the matrices over the unitization of the algebra

$$\mathcal{C}_n := B^{\frac{n}{n+1}}_{n+1,n+1}(\mathcal{M}) \cap \mathcal{M} = \{a \in \mathcal{M} \cap L^{n+1}(\mathcal{M}) : \hat{H}_a \in L^{n+1}(\mathcal{N})\}.$$

It is a $*$-algebra by the Leibniz property of the commutator and a consequence of Proposition 2.1.5 that the commutators of such symbols with $\mathbf{F}_{x_0}$ are also in $L^{n+1}(\mathcal{N})$ since the function $\lambda \in \hat{G} \mapsto \operatorname{sgn}(\gamma \cdot \lambda) - \operatorname{sgn}(\gamma \cdot (\lambda + x_0))$ lies in $L^{n+1}(\hat{G})$.

Thus the triple $(\mathcal{C}_n(\mathcal{M}), \mathcal{N}, \mathbf{F}_{x_0})$ defines a so-called $(n+1)$-summable semi-finite Fredholm module (see [35] for a formal definition). Moreover, the Fredholm module is even in even dimensions, since the matrix $\gamma_0 = (-\imath)^{\lfloor n/2 \rfloor} \gamma_1 \dots \gamma_n$ commutes with $\pi(\mathcal{C}_n(\mathcal{M}))$ (by definition since we identify $\pi = \mathbb{1}_N \otimes \pi$) and anticommutes with $\mathbf{F}_{x_0}$. In the grading induced by γ_0 one then has

$$\mathbf{F}_{x_0} = \begin{pmatrix} 0 & \mathbf{G}^*_{x_0} \\ \mathbf{G}_{x_0} & 0 \end{pmatrix}.$$

Computation of indices via the Fedosov-Calderon formula can be done via the some cyclic cocycles that are canonically associated to any $(n+1)$-summable Fredholm module [39]:

Definition 2.3.1 *The Chern cocycle for the bounded Fredholm modules given by $\mathbf{F}_{x_0}$ is a cyclic cocycle on the algebra $\mathcal{C}_n(\mathcal{M})$ defined for any $x_0 \in \mathbb{R}^n \setminus \mathbb{Z}^n$ by*

$$\begin{aligned} \widetilde{Ch}_{\mathcal{T},\alpha,x_0}(a_0, \dots, a_n) &= \tilde{c}_n\, \hat{\mathcal{T}}'_{\alpha,x_0}\big(\pi(a_0)[\mathbf{F}_{x_0}, \pi(a_1)] \cdots [\mathbf{F}_{x_0}, \pi(a_n)]\big) \\ &= \frac{\tilde{c}_n}{2}\, \hat{\mathcal{T}}_\alpha\big(\gamma_0 \mathbf{F}_{x_0}[\mathbf{F}_{x_0}, \pi(a_0)][\mathbf{F}_{x_0}, \pi(a_1)] \cdots [\mathbf{F}_{x_0}, \pi(a_n)]\big) \end{aligned} \tag{2.3.1}$$

with normalization constants

$$\tilde{c}_n \;=\; \begin{cases} (-1)^k\,, & \text{for } n = 2k\,,\\ (-1)^{k+1}\,2^{-2k-1}\,, & \text{for } n = 2k+1\,, \end{cases}$$

and with the supertrace

$$\hat{\mathcal{T}}'_{\alpha,x_0}(a) \;=\; \frac{1}{2}\,\hat{\mathcal{T}}_\alpha\big(\mathbf{F}_{x_0}\,d_{x_0}(\gamma_0 a)\big)\,,$$

the graded commutator $d_{x_0}a = \mathbf{F}_{x_0}a - (-1)^n a\mathbf{F}_{x_0}$ *and* $\gamma_0 = (-\imath)^{\lfloor n/2 \rfloor}\gamma_1\cdots\gamma_n$ *for n even respectively* $\gamma_0 = 1$ *for n odd such that* $\gamma_0\mathbf{F}_{x_0} + (-1)^n\mathbf{F}_{x_0}\gamma_0 = 0$. *Furthermore, the integrated cocycle is*

$$\widetilde{Ch}_{\mathcal{T},\alpha}(a_0,\dots,a_n) \;=\; \int_{[0,1]^n} \widetilde{Ch}_{\mathcal{T},\alpha,x_0}(a_0,\dots,a_n)\;\mathrm{d}^n x_0\,. \qquad (2.3.2)$$

Since $\hat{\mathcal{T}}_\alpha$ is invariant under the dual action of $\hat{G} = \mathbb{Z}^{n_0}\oplus\mathbb{R}^{n_1}$ the offset x_0 is irrelevant in the case of $n_0 = 0$ since the cocycles $\widetilde{\mathrm{Ch}}_{\mathcal{T},\alpha,x_0} = \widetilde{\mathrm{Ch}}_{\mathcal{T},\alpha}$ all coincide. For $n_0 > 0$ they are not identical, the purpose of the average is to remove the dependence on the base point.

If $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{M}$ in which $\mathcal{C}_n(\mathcal{M})\cap\mathcal{M}$ is norm-dense, then there are the index pairings

$$\langle[\mathbf{F}_{x_0}]_1,[u]_1\rangle = \hat{\mathcal{T}}_\alpha\text{−Ind}(\mathbf{P}_{x_0}\pi(u)\mathbf{P}_{x_0} + \mathbb{1} - \mathbf{P}_{x_0}) \qquad (2.3.3)$$

with $K_1(\mathcal{A})$ in the odd and

$$\langle[\mathbf{F}_{x_0}]_0,[e]_0 - [s(e)]_0\rangle = \hat{\mathcal{T}}_\alpha\text{−Ind}(\pi(e)\mathbf{G}_{x_0}\pi(e) + \mathbb{1} - \pi(e)) \qquad (2.3.4)$$

with $K_0(\mathcal{A})$ in the even case. Here $\mathbf{P}_{x_0} = \frac{\mathbb{1}+\mathbf{F}_{x_0}}{2}$ and one has representatives $u, p \in M_{N'}(\mathcal{A}^\sim)$ of $K_{n \bmod 2}(\mathcal{A})$. Technically we use the extension of the trace to $N' = NM$-dimensional matrices but that is notationally suppressed. Note that in particular the indices are well-defined since the compactness of the commutators with $\mathbf{F}_{x_0}$ extends to the C^*-closure $\mathcal{A}$. The index does not depend on the offset $x_0 \in (0,1]^n$ due to norm-continuous homotopy.

For $(n+1)$-summable symbols computation of the index is then standard using Theorem 2.2.2:

Proposition 2.3.2 ([111, **Proposition 3.4.8**]) *If n is odd and $u \in M_{N'}(\mathcal{C}_n(\mathcal{M})^{\sim})$ a unitary with scalar part $s(u)$ and hence $u - s(u) \in M_{N'}(\mathcal{C}_n(\mathcal{M}))$, then*

$$\hat{\mathcal{T}}_\alpha\text{-Ind}(\mathbf{P}_{x_0}\pi(u)\mathbf{P}_{x_0} + 1 - \mathbf{P}_{x_0}) \;=\; \widetilde{Ch}_{\mathcal{T},\alpha,x_0}(u^* - s(u^*), \dots, u - s(u)) \;,$$

with $\mathbf{P}_{x_0} = \frac{1+\mathbf{F}_{x_0}}{2}$ for any $x_0 \in \mathbb{R}^n$. If n is even and $e \in M_N(\mathcal{C}_n(\mathcal{M}))^{\sim}$ a projection with $e - s(e) \in M_N(\mathcal{C}_n(\mathcal{M}))$, then

$$\hat{\mathcal{T}}_\alpha\text{-Ind}(\pi(e)\mathbf{G}_{x_0}\pi(e) + 1 - \pi(e)) \;=\; \widetilde{Ch}_{\mathcal{T},\alpha,x_0}(e - s(e), \dots, e - s(e)) \;,$$

with $\mathbf{G}_{x_0}$ defined by $\mathbf{F}_{x_0} = \begin{pmatrix} 0 & \mathbf{G}_{x_0}^* \\ \mathbf{G}_{x_0} & 0 \end{pmatrix}$ in a basis such that $\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and any $x_0 \in (0,1]^n$.

Since the index almost-surely does not depend $x_0 \in (0,1]^n$ one can also average over the right-hand side, thus compute the index using the cocycle $\widetilde{\mathrm{Ch}}$. A major problem with the cocycles $\widetilde{\mathrm{Ch}}$ is, however, that it they not suited for practical computations, whether analytical or numerical evaluation using discretizations or finite-dimensional approximation. In practice our primary definition is therefore based on differentials in L^p-norms:

Definition 2.3.3 *Let $\nabla_1, \dots, \nabla_n$ be the derivations (1.4.1) on $W^1_n(\mathcal{M}, \alpha)$ w.r.t. the unit directions of G. Also let $p \in [n, n+1]$. The Chern cocycle for the action α is a cyclic cocyle on $\mathcal{M} \cap W^1_p(\mathcal{M})$ defined by*

$$Ch_{\mathcal{T},\alpha}(a_0, \dots, a_n) \;=\; c_n \sum_{\rho \in S_n} (-1)^\rho \, \mathcal{T}\big(a_0 \nabla_{\rho(1)} a_1 \dots \nabla_{\rho(n)} a_n\big) \,, \tag{2.3.5}$$

where the normalization constants are given by

$$c_n \;=\; \begin{cases} \frac{(2\pi\imath)^k}{k!} \,, & \text{for } n = 2k \,, \\ \frac{\imath\,(\pi\imath)^k}{(2k+1)!!} \,, & \text{for } n = 2k+1 \,. \end{cases}$$

The cocycle properties are rather easily checked on a dense subalgebra consisting of elements that are smooth w.r.t. the L^p-norm such that one can freely use the partial integration identity $\mathcal{T}(\nabla(a)b) = -\mathcal{T}(a\nabla(b))$. The cocycle is well-defined for any $p \in [n, n+1]$ and there do exist relevant examples where Sobolev regularity holds at neither endpoint of the interval $[n, n+1]$ but for all values inbetween, hence there is not always a canonical choice.

Nevertheless, $\mathrm{Ch}_{\mathcal{T},\alpha}$ is for each $p \in [n, n+1]$ a natural n-cocycle that is constructed from the data $\mathcal{T}$ and α. It is therefore not surprising (though a complicated computation) that it coincides with the other natural cocycle:

Proposition 2.3.4 ([111, Proposition 3.4.7]) *For $a_0, \dots, a_n \in W_p^1(\mathcal{M}) \cap \mathcal{C}_n(\mathcal{M})$, $p \in (n, n+1]$, one has*

$$Ch_{\mathcal{T},\alpha}(a_0, \dots, a_n) \;=\; (-1)^{n-1}\, \widetilde{Ch}_{\mathcal{T},\alpha}(a_0, \dots, a_n)\,.$$

The main result of this section, or rather of [111, Section 3.5], is that those Chern numbers can be written in terms of a semifinite index theorem:

Theorem 2.3.5 ([111, Sobolev index theorem 3.4.9]) *If n is odd and $u \in M_{N'}(\mathcal{M})$ a unitary in $M_{N'}(W_p^1(\mathcal{M})^\sim)$ for some $p \in (n, n+1]$, then*

$$Ch_{\mathcal{T},\alpha}(u^*, u, \dots, u^*, u) \;=\; -\,\hat{\mathcal{T}}_\alpha\text{-}\mathrm{Ind}(\mathbf{P}_{x_0}\pi(u)\mathbf{P}_{x_0} + 1 - \mathbf{P}_{x_0})\,,$$

with $\mathbf{P}_{x_0} = \frac{1}{2}(1 + \mathbf{F}_{x_0})$ for any $x_0 \in (0,1)^n$. If n is even and $e \in M_{N'}(\mathcal{M})$ a projection in $M_{N'}(W_p^1(\mathcal{M})^\sim)$ for some $p \in [n, n+1]$, then

$$Ch_{\mathcal{T},\alpha}(e, \dots, e) \;=\; \hat{\mathcal{T}}_\alpha\text{-}\mathrm{Ind}(\pi(e)\mathbf{G}_{x_0}\pi(e) + 1 - \pi(e))\,,$$

for any $x_0 \in (0,1)^n$ with $\mathbf{G}_{x_0}$ defined by $\mathbf{F}_{x_0} = \begin{pmatrix} 0 & \mathbf{G}_{x_0}^* \\ \mathbf{G}_{x_0} & 0 \end{pmatrix}$ in a basis such that $\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

This is just the combination of Proposition 2.3.2 and 2.3.4. Let us note that the regularity conditions simplify since $W_p^1(\mathcal{M}) \cap \mathcal{M} \subset \mathcal{C}_n(\mathcal{M})$ holds for $p \in (n, n+1]$ by Proposition 1.4.6 and for $p = n$ if $n \geq 2$. Thus the required Besov regularity is only ever in question for the extreme case $p = 1 = n$. That will rarely be an issue in applications.

2.4 Pairings with multipliers

Let $\mathcal{A} \subset \mathcal{M}$ be a C^*-algebra with $[\mathcal{A}, \mathbf{F}_{x_0}] \subset \mathcal{K}(\mathcal{N})$ as in the previous section, then there are the index pairings (2.3.3) and (2.3.4). Both are defined in terms of representatives in the standard picture of K-theory. For a representative in the multiplier picture one can immediately write down the odd pairing

$$\langle [\mathbf{F}_{x_0}]_1, [u_+]_1^M - [u_-]_1^M \rangle = \hat{\mathcal{T}}_\alpha\text{–Ind}(\mathbf{P}_{x_0}\pi(u_+u_-^*)\mathbf{P}_{x_0} + \mathbb{1} - \mathbf{P})$$

based on the canonical standard picture representative $[u_+]_1^M - [u_-]_1^M = [u_+u_-^*]_1$ and compute the index using one of the Chern cocycles. The required Sobolev regularity of $u_+u_-^*$ holds under a very simple sufficient condition:

Lemma 2.4.1 *Let $u_+, u_- \in \mathcal{M}$ be unitaries with u_- weakly differentiable and for which $u_+ - u_- \in W_p^1(\mathcal{M})$ then $u_+u_-^* - \mathbb{1} \in W_p^1(\mathcal{M})$.*

Proof. Clearly $u_+u_-^* - \mathbb{1} = (u_+ - u_-)u_-^* \in L^p(\mathcal{M}) \cap \mathcal{M}$. For a weakly differentiable element the difference quotients converge in the weak-$*$-topology with norm-bounded derivatives. Thus the weakly differentiable elements are two-sided multipliers of $W_p^1(\mathcal{M})$. In particular

$$\nabla_w(u_+u_-^*) = \nabla_w(u_+ - u_-)u_-^* + (u_+ - u_-)\nabla_w(u_-^*)$$

is in $L^p(\mathcal{M})$. □

For $\mathcal{A}$ the C^*-completion of $\mathcal{M} \cap W_p^1(\mathcal{M})$ for some $p \in [n, n+1]$ with n even one has a pairing with classes in the multiplier picture

$$\langle [\mathbf{F}_{x_0}]_0, [e_+]_0^M - [e_-]_0^M \rangle$$

which is obviously defined by representing the class $[e_+]_0^M - [e_-]_0^M \in K_0(\mathcal{A})$ in the standard picture. While a representative in standard form $[e]_0 - [s(e)]_0$ with $e \in M_N(W_p^1(\mathcal{M})^\sim)$ must exist, there is apparently no simple general algebraic expression to determine it. In practice this also means there is no feasible way to compute the pairing using an even-dimensional Chern cocycle. An easy and well-known way to avoid that problem is to suspend the cocycle to the algebra $\overline{S}\mathcal{M} := L^\infty(\mathbb{R}) \otimes \mathcal{M}$ (with W^*-algebraic tensor product) using Theorem 1.6.3:

Proposition 2.4.2 *Let n be even, $e_+, e_- \in M_N(\mathcal{M})$ be projections with $e_+ - e_- \in M_N(W_p^1(\mathcal{M}))$ for some $p \in [n, n+1]$ and e_- weakly differentiable. Then $[e_+]_0^M -$*

$[e_-]_0^M$ determines a class in $K_0(\mathcal{A})$ for the C^*-algebra $\mathcal{A} = C^*(\mathcal{M} \cap W_p^1(\mathcal{M}))$. one has a well-defined index pairing

$$\langle [\mathbf{F}_{x_0}]_0, [e_+]_0^M - [e_-]_0^M \rangle.$$

Let $f \in \mathcal{S}(\mathbb{R})^\sim$ be the function $f(\lambda) = \frac{\sinh(\lambda)-\imath}{\sinh(\lambda)+\imath}$ of winding number 1 and define the unitaries $u_\pm = f \otimes e_\pm + 1 \otimes e_\pm \in (\overline{S}\mathcal{M})^2$. One has

$$\begin{aligned}\langle [\mathbf{F}_{x_0}]_0, [e_+]_0^M - [e_-]_0^M \rangle &= \langle Ch_{\mathcal{T},\alpha}, [e_+]_0^M - [e_-]_0^M \rangle \\ &= \langle Ch_{\mathcal{T}^s,\alpha\times\lambda}, [\hat{u}_+ \hat{u}_-^*]_1 \rangle\end{aligned}$$

where $\mathcal{T}^s = \int_{\mathbb{R}} \mathrm{d}t \otimes \mathcal{T}$ with integration w.r.t. the Lebesgue measure and $\lambda : L^\infty(\mathbb{R}) \times \mathbb{R} \to L^\infty(\mathbb{R})$ left translation.

Proof. Since weakly differentiable elements preserve $W_p^1(\mathcal{M}) \cap \mathcal{M}$ under left and right multiplication one finds that e_-, e_+ are naturally included into $M^s(\mathcal{A})$ and hence the class $[e_+]_0^M - [e_-]_0^M \in K_0(\mathcal{A})$ is well-defined. Let $[e]_0 - [s(e)]_0$ a representative in standard form with $e \in M_N(W_p^1(\mathcal{M})^\sim)$ which exists due to spectral invariance. With the suspension $\hat{u} = (f \otimes e + 1 \otimes e)(\overline{f} \otimes s(e) + 1 \otimes s(e))$ Theorem 1.6.3 implies

$$\begin{aligned}\langle [\mathbf{F}_{x_0}]_0, [e_+]_0^M - [e_-]_0^M \rangle &= \langle \mathrm{Ch}_{\mathcal{T},\alpha}, [e]_0 - [s(e)]_0 \rangle \\ &= \langle \mathrm{Ch}_{\mathcal{T}^s,\alpha\times\lambda}, [\hat{u}]_1 \rangle\end{aligned}$$

with slight abuse of notation since the suspension cocycle $(\mathrm{Ch}_{\mathcal{T},\alpha})^s$ while given by the same formal expression (2.3.5) has a different domain than $\mathrm{Ch}_{\mathcal{T}^s,\alpha\times\lambda}$, namely one requires spaces of mixed smoothness; the derivatives with respect to α lie in $L^p(\overline{S}\mathcal{M})$ while those with respect to the suspension variable lie in $L^p(\overline{S}\mathcal{M}) \cap L^\infty(\overline{S}\mathcal{M})$. The suspension $\hat{u}$ lies in that domain and $\hat{u}_+\hat{u}_-^*$ as well and since both represent the same class in $K_1(S\mathcal{A})$ the proof is finished. □

In particular the suspension is immediately regular enough to compute the suspended cocycle. However, one cannot directly apply the index theorem since that requires regularity in $B_{n+2,n+2}^{(n+1)/(n+2)}(\overline{S}\mathcal{M})$ which is stronger than the input of $W_p^1(\mathcal{M})$. For an index theorem with an explicitly computable Fredholm operator we must therefore ask for more regularity which is often not an issue:

Corollary 2.4.3 *Let n be even and $e_+, e_- \in M_N(\mathcal{M})$ be projections with $e_+ - e_- \in M_N(W^1_{p_1}(\mathcal{M}) \cap W^1_{p_2}(\mathcal{M}))$ for some $p_1 \in [n, n+1]$, $p_2 \in [n+1, n+2]$and e_- weakly differentiable. Then*

$$\langle [\mathbf{F}_{x_0}]_0, [e_+]^M_0 - [e_-]^M_0 \rangle = \widehat{(\mathcal{T}^s)}_{\alpha\times\lambda}\text{-Ind}(\tilde{\mathbf{P}}_{x_0}\pi(u_+u_-^*)\tilde{\mathbf{P}}_{x_0} + 1 - \tilde{\mathbf{P}}_{x_0})$$

where $\tilde{\mathbf{P}}_{x_0} = \chi(\mathbf{D} + \gamma X) > 0$ with X the multiplication operator by the function id *on $L^\infty(\mathbb{R})$ and $\gamma = (-\iota)^{\lfloor n/2 \rfloor}\gamma_0 ... \gamma_{d-1}$.*

Proof. Lemma 2.4.1 implies that $\hat{u}_+\hat{u}_- \in M_N(\overline{S}\mathcal{M} \cap W^1_{p_2}(\mathcal{M})^\sim)$ which is regular enough for the Sobolev index theorem and the value of the cocycle obviously does not depend on the domain. □

Instead of the suspension to $L^\infty(\mathbb{R}) \otimes \mathcal{M}$ one could as well have suspended to $L^\infty(\mathbb{T}) \otimes \mathcal{M}$, the Chern cocycles on the suspension are numerically equal up to a reparametrization.

3 Toeplitz extensions for one-parameter actions

A common situation in index theory is that one deals with a C^*-algebra $\mathcal{A}$ and studies an extension containing the C^*-algebraic span of elements $P\pi(\mathcal{A})P$ in a representation π with P a projection that is not a multiplier of $\mathcal{A}$. An important special case is that of a C^*-algebra with an $\mathbb{R}$-action where P is the spectral projection of the generator in a covariant representation [70][81][96]. Then the obtained extension may be called the Toeplitz extension associated to a flow since it is generated by operators of the form $P\pi(a)P$ exactly as the discrete Toeplitz extension that e.g. features in the Pimsner-Voiculescu sequence. The K-theory of such a Toeplitz extension can be very difficult to compute. By modifying the construction to use instead of a sharp spectral projection a continuous switch function one obtains a much more accessible algebra, the so-called smoothed Toeplitz extension [70], which is intimately related to crossed products and offers very simple connecting maps in K-theory.

Toeplitz extensions play an important role in K-theory but also in solid state physics where they feature prominently as bulk-boundary exact sequences that link phenomena on the surface of a system with those in its bulk. From that arises also the need for an analogue of the smoothed Toeplitz extension for an action of the torus. In [111, Chapter 4] those two versions were treated on the same footing as the so-called smooth Toeplitz extension for a one-parameter group. In this section we recall the construction and its properties for use in Chapter 5. Some related algebras and exact sequences are also introduced for use in bulk-interface correspondence. In the final section we explore a variant of the extensions based on stable multipliers.

3.1 The smooth one- and two-sided Toeplitz extension

In this chapter let $(\mathcal{A}, G, \xi)$ be a C^*-dynamical system with a one-parameter group G, i.e. $G = \mathbb{T}$ or $G = \mathbb{R}$. We assume that $\mathcal{A}$ acts faithfully and non-degenerately on a Hilbert space $\mathcal{H}_0$ and identify $\mathcal{A} \rtimes_\xi G$ with its image in the regular representation $\pi \times U$ on $\mathcal{H} = L^2(G, \mathcal{H}_0)$. The action ξ is implemented by exponentiation of the self-adjoint generator D as above in (2.1.1). One can then write $\mathcal{A} \rtimes_\xi G$ as then the C^*-algebraic span of

$$\mathcal{A} \rtimes_\xi G \;=\; C^*\left\{\pi(a)f(D) \;:\; a \in \mathcal{A},\; f \in C_0(\hat{G})\right\} \;\subset\; \mathcal{B}(\mathcal{H})\,.$$

Here $\hat{G} = \mathbb{Z}$, or $\hat{G} = \mathbb{R}$. Let $C_{0,*}(\hat{G})$ be the continuous functions which vanish in $-\infty$ and admit a limit in $+\infty$, respectively $C_{*,0}(\hat{G})$ those who vanish in $+\infty$ and admit a limit in $-\infty$. Similarly let $C_{*,*}(\hat{G})$ be the functions which admit finite, but possibly different limits at $\pm\infty$. The smooth Toeplitz extensions are obtained by supplementing the crossed product algebras with further functions of the generator:

Definition 3.1.1 ([81, 70]) *The smooth Toeplitz extensions associated to a C^*-dynamical system $(\mathcal{A}, G, \xi)$ as above are defined by*

$$T_-(\mathcal{A}, \xi, G) \;=\; C^*\left\{\pi(a)f(D) \;:\; a \in \mathcal{A},\; f \in C_{*,0}(\hat{G})\right\} \;\subset\; \mathcal{B}(\mathcal{H})\,. \tag{3.1.1}$$

or

$$T_+(\mathcal{A}, \xi, G) \;=\; C^*\left\{\pi(a)f(D) \;:\; a \in \mathcal{A},\; f \in C_{0,*}(\hat{G})\right\} \;\subset\; \mathcal{B}(\mathcal{H})\,. \tag{3.1.2}$$

or

$$T(\mathcal{A}, \xi, G) \;=\; C^*\left\{\pi(a)f(D) \;:\; a \in \mathcal{A},\; f \in C_{*,*}(\hat{G})\right\} \;\subset\; \mathcal{B}(\mathcal{H})\,. \tag{3.1.3}$$

and are called the left, right or two-sided Toeplitz extension respectively.

It is easy to see that the left and right extension are canonically isomorphic to each other by inverting the action $\xi \to \xi^{-1}$.

Let Θ be a continuous switch function converging to 1 at $+\infty$ and 0 at $-\infty$. It is sometimes convenient to assume further that Θ' is compactly supported. The functions $\mathcal{P}_+ = \Theta(D)$ and $\mathcal{P}_- = 1-\Theta(D)$ are multipliers of $\mathcal{A}\rtimes_\xi G$. For consistency one needs to show that the Toeplitz extensions arise by complementing the crossed product with elements $\mathcal{P}_+\pi(a)$, $\mathcal{P}_-\pi(a)$ and $\mathcal{P}_+\pi(a_+) + \mathcal{P}_-\pi(a_+)$. This is well-known for the right Toeplitz extension [81, 70] and for completeness we lift the proof of [111, Lemma 4.1.2] to this slightly more general setting:

Lemma 3.1.2 *Let $a \in \mathcal{A}$ and $\mathcal{P}_+$ be constructed as above. Then $[\mathcal{P}_+, \pi(a)] \in \mathcal{A} \rtimes_\xi G$ and hence $\mathcal{A} \rtimes_\xi G$ is a two-sided ideal in $T(\mathcal{A}, G, \xi)$. Furthermore, the elements of $T(\mathcal{A}, G, \xi)$ are precisely those operators which can be written in the*

$$\hat{a} = \pi(a_+)\mathcal{P}_+ + \pi(a_-)\mathcal{P}_- + \hat{e}$$

with unique $a_\pm \in \mathcal{A}$, $\hat{e} \in \mathcal{A} \rtimes_\xi G$.

Proof. For the first part it is enough to show that $[\mathcal{P}_+, \pi(a)] \in \mathcal{A} \rtimes_\xi G$ for $a \in C^\infty(\mathcal{A})$, the dense subset of norm-smooth elements w.r.t. ξ. By Proposition A.5 one has as a norm-convergent Riemann integral

$$[\mathcal{P}_+, a] = \frac{1}{2\pi} \int_{\mathbb{C}} (\partial_{\overline{z}} \tilde{\Theta}_K)(z) \frac{1}{D - z} [a, D] \frac{1}{D - z} \mathrm{d}z \wedge \mathrm{d}\overline{z}$$

for a suitable function $\tilde{\Theta}_K$. Since $[a, D] \in \mathcal{A}$ the integrand lies in $\mathcal{A} \rtimes_\xi G$ pointwise, hence $[\mathcal{P}_+, a]$.

That implies that $\mathcal{A} \rtimes_\xi G$ is an ideal in $\mathrm{T}(\mathcal{A}, G, \xi)$ since, e.g.,

$$\Theta(D)(\pi(a) f(D)) \;=\; [\Theta(D), \pi(a)] f(D) + \pi(a)(\Theta f)(D) \;\in\; \mathcal{A} \rtimes_\xi G\,,$$

for all $a \in \mathcal{A}$, $f \in C_0(\hat{G})$ and $\Theta \in C_{0,*}(\hat{G})$ due to $\Theta f \in C_0(\hat{G})$.

It also follows that the linear span of elements

$$\hat{a} = \pi(a_+)\mathcal{P}_+ + \pi(a_-)\mathcal{P}_- + \sum_{i=1}^{K} f_i(D)\pi(a_i) \tag{3.1.4}$$

with $f_i \in C_c(\hat{G})$, $a_i \in \mathcal{A}$ is already norm-dense in $\mathrm{T}(\mathcal{A}, G, \xi)$.

Let $\hat{\xi}$ be the dual action, then it is not difficult to see in the standard regular representation that $q_\pm(\hat{a}) := \text{s-lim}_{t \to \mp\infty} \hat{\xi}_t(\hat{a}) = a_\pm$ for each such $\hat{a}$. Moreover, one can easily show that

$$\left\| \pi(a_+)\mathcal{P}_+ + \pi(a_-)\mathcal{P}_- + \sum_{i=1}^{K} f_i(D)\pi(a_i) \right\| \geq \max(\|a_+\|, \|a_-\|)$$

with any $a_i \in \mathcal{A}$, $f_i \in C_c(\mathbb{R})$ by translating trial vectors to $\pm\infty$ (see the proof of [111, Lemma 4.1.2]). One concludes that the norm-closure of elements of the form

(3.1.4) must be equal to $\mathcal{A}\mathcal{P}_+ + \mathcal{A}\mathcal{P}_- + \mathcal{A} \rtimes_\xi G$ since a sequence of elements of form (3.1.4) can only converge in norm if all three terms converge individually. Hence

$$\mathrm{T}(\mathcal{A}, G, \xi) = \mathcal{A}\mathcal{P}_+ + \mathcal{A}\mathcal{P}_- + \mathcal{A} \rtimes_\xi G$$

and the map $q = (q_-, q_+) : \mathrm{T}(\mathcal{A}, G, \xi) \to \mathcal{A} \oplus \mathcal{A}$ is a surjective homomorphism with kernel $\mathcal{A} \rtimes_\xi G$. □

This immediately implies:

Proposition 3.1.3 *There are the exact sequences*

$$0 \to \mathcal{A} \rtimes_\xi G \hookrightarrow T_\pm(\mathcal{A}, \xi, G) \overset{q_\pm}{\to} \mathcal{A}_\pm \to 0\,. \tag{3.1.5}$$

and

$$0 \to \mathcal{A} \rtimes_\xi G \hookrightarrow T(\mathcal{A}, \xi, G) \overset{q}{\to} \mathcal{A}_- \oplus \mathcal{A}_+ \to 0\,. \tag{3.1.6}$$

where $\mathcal{A}_\pm = \mathcal{A}$ are two copies of $\mathcal{A}$.

Each of these Toeplitz extensions includes naturally into the multiplier algebra $M(\mathcal{A} \rtimes_\xi G)$, which makes it independent of the regular representation used for its construction. Furthermore, any representation of $\mathcal{A} \rtimes_\xi G$ gives rise to a representations of the Toeplitz algebras.

3.2 Connecting maps

This section aims to describe the connecting maps in K-theory induced by the Toeplitz extensions. As the primary object of study we consider the maps for $\mathrm{T}_+(\mathcal{A}, \xi, G)$ which will be denoted by

$$\mathrm{Ind}_G^\xi : K_1(\mathcal{A}) \to K_0(\mathcal{A} \rtimes_\xi G)\,, \qquad \mathrm{Exp}_G^\xi : K_0(\mathcal{A}) \to K_1(\mathcal{A} \rtimes_\xi G)\,.$$

The connecting maps for $\mathrm{T}_-(\mathcal{A}, \xi, G)$ are obtained via a natural isomorphism

$$\Sigma : \mathcal{A} \rtimes_\xi G \to \mathcal{A} \rtimes_{\xi^{-1}} G$$

which extends to a map which flips the sign of the action, for $\mathrm{T}(\mathcal{A}, \xi, G)$ the exponential map is then

$$(\mathrm{Exp}_G^\xi) \circ (\pi_+)_* + \Sigma_* \circ (\mathrm{Exp}_G^{\xi^{-1}}) \circ (\pi_-)_* = (\mathrm{Exp}_G^\xi) \circ ((\pi_+)_* - (\pi_-)_*)$$

and the index map

$$(\mathrm{Ind}_G^{\xi}) \circ (\pi_+)_* + \Sigma_* \circ (\mathrm{Ind}_G^{\xi^{-1}}) \circ (\pi_-)_* = (\mathrm{Ind}_G^{\xi}) \circ ((\pi_+)_* - (\pi_-)_*)$$

with $\pi_\pm : \mathcal{A}_+ \oplus \mathcal{A}_- \to \mathcal{A}$.

Since the smooth Toeplitz extension is constructed naturally and functorially in terms of $\mathcal{A}$ and α it is not difficult to verify that its connecting maps satisfy the axiomatic characterization of the Connes-Thom isomorphisms [37]:

Proposition 3.2.1 ([111, Proposition 4.2.3]) *The connecting maps of the smooth Toeplitz extension $T_+(\mathcal{A}, \xi, \mathbb{R})$ written as $\mathrm{Exp}_{\mathbb{R}}^{\xi}$ and $\mathrm{Ind}_{\mathbb{R}}^{\xi}$ are related to the Connes-Thom isomorphisms by*

$$\mathrm{Exp}_{\mathbb{R}}^{\xi} \; = \; -(\partial_{\mathcal{A}})_0^{\xi}\,, \qquad (\mathrm{Ind}_{\mathbb{R}})^{\xi} \; = \; -(\partial_{\mathcal{A}})_1^{\xi}\,.$$

In particular they are isomorphisms and replacing ξ with ξ^{-1} one indeed has $\mathrm{Exp}_{\mathbb{R}}^{\xi} = -\Sigma_* \circ \mathrm{Exp}_{\mathbb{R}}^{\xi^{-1}}$ due to the way in which a choice of orientation of $\mathbb{R}$ enters into the axiomatic characterization of the Connes-Thom isomorphism. In the case $G = \mathbb{T}$ we can first apply the result for ξ considered as an $\mathbb{R}$-action and then factor:

Proposition 3.2.2 ([111, Proposition 4.2.6]) *Consider a given $\mathbb{T}$-action ξ on $\mathcal{A}$ as a $\mathbb{R}$-action by setting $\xi_{t+1} = \xi_t$. Then the connecting maps of $T_+(\mathcal{A}, \xi, \mathbb{R})$ are given by*

$$\mathrm{Ind}_{\mathbb{T}}^{\xi} \; = \; \Pi_* \circ \mathrm{Ind}_{\mathbb{R}}^{\xi}\,, \qquad \mathrm{Exp}_{\mathbb{T}}^{\xi} \; = \; \Pi_* \circ \mathrm{Exp}_{\mathbb{R}}^{\xi}\,, \tag{3.2.1}$$

with $\Pi : \mathcal{A} \rtimes_\xi \mathbb{R} \to \mathcal{A} \rtimes_\xi \mathbb{T}$ the natural surjection defined by

$$\Pi\left(\int_{\mathbb{R}} f(x) e^{2\pi\imath D_{\mathbb{R}} x}\,\mathrm{d}x\right) \; = \; \int_{\mathbb{T}} \left(\sum_{k\in\mathbb{Z}} f(x+k)\right) e^{2\pi\imath D_{\mathbb{T}} x}\,\mathrm{d}x$$

for all $f \in C_c(\mathbb{R}, \mathcal{A})$ with $D_{\mathbb{R}}$ and $D_{\mathbb{T}}$ generators of the respective actions in a faithful representation of the respective crossed product.

While the connecting maps for $G = \mathbb{R}$ are always isomorphisms this is rarely the case for $G = \mathbb{T}$ as one can already convince oneself from the case where ξ is the trivial action.

The usual (discrete) Toeplitz extension and the Wiener-Hopf extension are special cases of the smooth Toeplitz extension with the dual action:

Lemma 3.2.3 ([111, Lemma 4.2.7], [70]) *For a C^*-algebra $\mathcal{B}$ with $\mathbb{Z}$-action β, the exact sequence*

$$0 \;\to\; C_0(\mathbb{Z},\mathcal{B}) \rtimes_{\lambda\otimes\beta} \mathbb{Z} \;\hookrightarrow\; C_{0,*}(\mathbb{Z},\mathcal{B}) \rtimes_{\lambda\otimes\beta} \mathbb{Z} \;\to\; \mathcal{B} \rtimes_\beta \mathbb{Z} \;\to\; 0$$

with right translation λ is naturally isomorphic to the smooth $\mathbb{T}$-Toeplitz extension

$$0 \;\to\; (\mathcal{B} \rtimes_\beta \mathbb{Z}) \rtimes_{\hat\beta} \mathbb{T} \;\hookrightarrow\; T_+(\mathcal{B} \rtimes_\beta \mathbb{Z}, \hat\beta, \mathbb{T}) \;\overset{\Pi}{\to}\; \mathcal{B} \rtimes_\beta \mathbb{Z} \;\to\; 0\,.$$

Likewise, for a C^*-algebra $\mathcal{B}$ with $\mathbb{R}$-action β, the exact sequence

$$0 \;\to\; C_0(\mathbb{R},\mathcal{B}) \rtimes_{\lambda\otimes\beta} \mathbb{R} \;\hookrightarrow\; C_{0,*}(\mathbb{R},\mathcal{B}) \rtimes_{\lambda\otimes\beta} \mathbb{R} \;\to\; \mathcal{B} \rtimes_\beta \mathbb{R} \;\to\; 0$$

isomorphic to the smooth $\mathbb{T}$-Toeplitz extension

$$0 \;\to\; (\mathcal{B} \rtimes_\beta \mathbb{R}) \rtimes_{\hat\beta} \mathbb{R} \;\hookrightarrow\; T_+(\mathcal{B} \rtimes_\beta \mathbb{R}, \hat\beta, \mathbb{R}) \;\overset{\Pi}{\to}\; \mathcal{B} \rtimes_\beta \mathbb{R} \;\to\; 0\,.$$

The connecting maps in the discrete case are therefore equivalent to those of the Pimsner-Voiculescu sequence[20, 10.2] and in the continuous to those of the Wiener-Hopf extension.

3.3 Chern numbers and duality

Let the C^*-algebra $\mathcal{A}$ be now equipped with a faithful densely defined lower semicontinuous trace $\mathcal{T}$ and an additional $\mathbb{R}^{n-1}$-action θ. Assume that θ and ξ commute and leave $\mathcal{T}$ invariant. Define $\alpha = \theta \times \xi$ as an $\mathbb{R}^n$-action.

Definition 3.3.1 *The Chern cocycle for the action α is a cyclic n-cocycle on the domain $(\mathcal{A}_{\mathcal{T},\alpha})^{n+1}$ defined by*

$$Ch_{\mathcal{T},\alpha}(a_0,\dots,a_n) \;=\; c_n \sum_{\rho\in S_n} (-1)^\rho\, \mathcal{T}\big(a_0 \nabla_{\rho(1)} a_1 \cdots \nabla_{\rho(n)} a_n\big)\,,$$

where $\nabla_1,\dots,\nabla_n$ are the derivations (1.4.1) on $\mathcal{A}_{\mathcal{T},\alpha}$ w.r.t. to an orthonormal basis $e_1,\dots,e_n$ of the Lie algebra $\mathbb{R}^n$ of G, S_n is the symmetric group and $(-1)^\sigma$ the signum *of a permutation $\sigma \in S_n$ and the normalization constants are as in Definition 2.3.3 given by*

$$c_n = \begin{cases} \frac{(2\pi\imath)^k}{k!}, & \text{for } n = 2k, \\ \frac{\imath(\pi\imath)^k}{(2k+1)!!}, & \text{for } n = 2k+1. \end{cases} \tag{3.3.1}$$

Here $\mathcal{A}_{\mathcal{T},\alpha}$ is the Fréchet algebra introduced in Section 1.4.1. The cocycles are given by the same expressions as the Chern cocycles in Chapter 2 but with different domains and notions of derivative.

For the smooth Toeplitz extension one has a duality result for the Chern cocycles:

Theorem 3.3.2 ([111, Theorem 4.5.3]) *Let $\mathcal{A}$ be a C^*-algebra with two commuting actions, namely a strongly continuous $\mathbb{R}^n$-action θ and a strongly continuous G-action ξ, where G is a one-parameter group. Let $\mathcal{T}$ be a densely defined faithful lower semicontinuous trace on $\mathcal{A}$ that is invariant under $\theta \times \xi$. With the connecting maps Ind_G^ξ and Exp_G^ξ of the smooth Toeplitz extension, one has*

$$\langle Ch_{\mathcal{T},\theta\times\xi}, [e]_0 - [s(e)]_0 \rangle = \langle Ch_{\hat{\mathcal{T}}_\xi,\theta}, \mathrm{Exp}_G^\xi([e]_0 - [s(e)]_0) \rangle$$

for n odd and $[e]_0 - [s(e)]_0 \in K_0(\mathcal{A})$, respectively

$$\langle Ch_{\mathcal{T},\theta\times\xi}, [v]_1 \rangle = -\langle Ch_{\hat{\mathcal{T}}_\xi,\theta}, \mathrm{Ind}_G^\xi [v]_1 \rangle$$

for n even and $[v]_1 \in K_1(\mathcal{A})$.

It is a generalization of similar duality results for the Wiener-Hopf extension [75, 52] and the discrete Toeplitz extension [91, 74, 103] which are special cases of the smooth Toeplitz extension as seen in Lemma 3.2.3. The proof is based on a natural duality for crossed products with $\mathbb{R}$: If $\mathscr{A}$ is a Fréchet subalgebra of $\mathcal{A}$ which is strongly spectrally invariant in the sense of Theorem 1.6.2 then one can construct a smooth crossed product $\mathscr{A} \rtimes_\xi \mathbb{R}$ which is spectrally invariant in $\mathcal{A} \rtimes_\xi \mathbb{R}$. If φ is a cyclic n-cocycle on $\mathscr{A}$ then one has a natural $(n+1)$-cycle $(\varphi_\xi, d^\xi, \mathcal{A} \rtimes_\xi \mathbb{R})$ on the crossed product, which pairs dually with respect to the Connes-Thom isomorphisms

$$\langle \#_\xi \varphi, (\partial_{\mathcal{A}})_i(x) \rangle = (-1)^i \langle \varphi, x \rangle$$

for $x \in K_{1-i}(\mathcal{A})$. The general formula is not important here, but assuming θ and ξ commute it reduces to

$$(\#_\xi \mathrm{Ch}_{\mathcal{T},\theta}) = \mathrm{Ch}_{\hat{\mathcal{T}}_\xi,\theta\times\hat{\xi}}$$

with the dual action $\hat{\xi}$. The Theorem above in the case of $\mathbb{R}$ follows by dualisation with Takai duality (see Theorem 1.1.6) since $\mathrm{Ch}_{(\hat{\mathcal{T}}_\xi)^\wedge_{\hat{\xi}},\theta\times\hat{\hat{\xi}}}$ and $\mathrm{Ch}_{\mathcal{T},\theta\times\xi}$ result in equivalent pairings with $K_i(\mathcal{A}\otimes_\xi \mathbb{R}\otimes_{\hat{\xi}} \mathbb{R}) \simeq K_i(\mathcal{A}\otimes\mathbb{K})$. In the case $G=\mathbb{T}$ one then applies Proposition 3.2.2 and uses that pairings of K-theory classes with Chern cocycles are unchanged under the map Π_*.

3.4 Multiplier Toeplitz extension

Motivated by our applications in physics we will have need for an analogue of the two-sided Toeplitz extension which connects two multipliers instead of two elements of $\mathcal{A}$. Such an algebra can be more or less naturally constructed as a subset of the multiplier algebra of two-sided Toeplitz extension, which is chosen small enough to have a well-behaved K-theory.

Proposition 3.4.1 *Let $(\mathcal{A},\xi,G)$ be a C^*-dynamical system with separable $\mathcal{A}$ then there is an exact sequence*

$$0 \to \mathcal{A}\rtimes_\xi G \to T^M(\mathcal{A},\xi,G) \to \mathbb{P}(M(\mathcal{A}),\mathcal{A}) \to 0$$

involving the C^*-algebra $T^M(\mathcal{A},\xi,G) = M(\mathcal{A}) + \mathcal{A}\mathcal{P}_+ + \mathcal{A}\rtimes_\xi G$.

Proof. The proof that $\mathrm{T}^M(\mathcal{A},\xi,G)$ is closed and a C^*-algebra can be done similarly to Lemma 3.1.2, since $m\mathcal{P}_+a = m[\mathcal{P}_+,a] + ma\mathcal{P}_+$ and $[\mathcal{A},\mathcal{P}_+]\subset \mathcal{A}\rtimes_\xi G$ as well as $M(\mathcal{A})\subset M(\mathcal{A}\rtimes_\xi G)$.

For the exact sequence we note that $\mathrm{T}^M(\mathcal{A},\xi,G)\subset M(\mathrm{T}(\mathcal{A},\xi,G))$ and there is a surjective extension $\hat{q}: M(\mathrm{T}(\mathcal{A},\xi,G)) \to M(\mathcal{A}_+\oplus\mathcal{A}_-) = M(\mathcal{A})\oplus M(\mathcal{A})$. For an element of $\mathrm{T}^M(\mathcal{A},\xi,G)$ one has $\hat{q}(m+a\mathcal{P}_+ + \hat{e}) = (m+a,m)\in\mathbb{P}(M(\mathcal{A}),\mathcal{A})$ and the kernel of $\hat{q}|_{\mathrm{T}^M(\mathcal{A},\xi,G)}$ is $\mathcal{A}\rtimes_\xi G$. □

It is important here that the two elements at infinity differ by only an element of $\mathcal{A}$ to constrain the K-theory. As an application we can extend the Toeplitz extension to multipliers in a way that interacts nicely with the multiplier picture of K-theory:

Proposition 3.4.2 *The multiplier Toeplitz extension $T^M(\mathcal{A}\otimes\mathbb{K},\xi,G)$ extends the Toeplitz extension to a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A}\rtimes_\xi G & \longrightarrow & T_+(\mathcal{A},\xi,G) & \xrightarrow{q} & \mathcal{A}_+ & \longrightarrow & 0 \\ & & \downarrow\rho & & \downarrow\rho & & \downarrow\rho_+ & & \\ 0 & \to & \mathcal{A}\rtimes_\xi G\otimes\mathbb{K} & \to & T^M(\mathcal{A}\otimes\mathbb{K},\xi,G) & \xrightarrow{\hat{q}} & \mathbb{P}(M^s(\mathcal{A}),\mathcal{A}\otimes\mathbb{K}) & \to & 0 \end{array} \tag{3.4.1}$$

with $\rho(x)=x\otimes e$ for some rank-one projection e and $\rho_+(x)=(x\otimes e,0)$. Moreover, both the left and the right vertical arrow induce isomorphisms in K-theory.

Proof. It is clear that the diagram commutes by construction of the map $\hat{q}$. The tensor product with a rank-one-projection implements the natural isomorphism $K_i(\mathcal{A}\rtimes_\xi G)\simeq K_i(\mathcal{A}\rtimes_\xi G\otimes\mathbb{K})$ and ρ_+ induces an isomorphism in K-theory by Lemma 1.5.3. □

Similarly, there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A}\rtimes_\xi G & \longrightarrow & \mathrm{T}(\mathcal{A},\xi,G) & \xrightarrow{q} & \mathcal{A}_-\oplus\mathcal{A}_+ & \longrightarrow & 0 \\ & & \downarrow\rho & & \downarrow\rho & & \downarrow\rho_-\oplus\rho_+ & & \\ 0 & \to & \mathcal{A}\rtimes_\xi G\otimes\mathbb{K} & \to & \mathrm{T}^M(\mathcal{A}\otimes\mathbb{K},\xi,G) & \xrightarrow{\hat{q}} & \mathbb{P}(M^s(\mathcal{A}),\mathcal{A}\otimes\mathbb{K}) & \to & 0 \end{array} \tag{3.4.2}$$

with $\rho_-(x)=(0,x\otimes e)$ which shows that the connecting maps are also equivalent to those of the two-sided Toeplitz extension. From that and Theorem 3.3.2 we immediately conclude

Corollary 3.4.3 *Consider the multiplier Toeplitz extension $T^M(\mathcal{A}\otimes\mathbb{K},\xi,G)$ in the case where $\mathcal{A}$ carries an additional $\mathbb{R}^n$-action θ that commutes with ξ as well as a $\theta\times\xi$-invariant densely defined lower semi-continuous trace $\mathcal{T}$.*

If n is odd and $(e_+,e_-)\in\mathbb{P}(M^s(\mathcal{A}),\mathcal{A}\otimes\mathbb{K})$ a projection then

$$\langle Ch_{\hat{\mathcal{T}}_\xi,\theta}, \mathrm{Exp}([(e_+,e_-)]_0)\rangle \;=\; \langle Ch_{\mathcal{T},\theta\times\xi}, [e_+]_0^M-[e_-]_0^M)\rangle$$

for the exponential map of the bottom exact sequence from Proposition 3.4.2.

Likewise, for n even and $(u_+,u_-)\in\mathbb{P}(M^s(\mathcal{A}),\mathcal{A}\otimes\mathbb{K})$ a unitary one has

$$\langle Ch_{\hat{\mathcal{T}}_\xi,\theta}, \mathrm{Ind}([(u_+,u_-)]_1)\rangle \;=\; -\langle Ch_{\mathcal{T},\theta\times\xi}, [u_+]_1^M-[u_-]_1^M\rangle$$

for the index map of the same exact sequence.

Thus we are lead to compute the Chern numbers of classes in $K_i(\mathcal{A})$ expressed by representatives in the multiplier picture. In the odd case those can be computed rather easily by going to the standard picture and in the even case one can use suspensions as it was done in Chapter 2. Here we give another approach which sometimes results in much simpler formulas. The idea is to define the Chern cocycles directly on a smooth subalgebra of $\mathbb{P}(M^s(\mathcal{A}), \mathcal{A} \otimes \mathbb{K})$.

Proposition 3.4.4 *Let $\mathcal{A}$ be a C^*-algebra with densely defined faithful lower-semicontinuous trace invariant under some $\mathbb{R}^n$-action θ. Then define*

$$\begin{aligned}&\mathscr{P}(\mathcal{A}, \mathcal{T}, \theta)\\ &\quad = \{(m, m+a) \in \mathbb{P}(M^s(\mathcal{A}), \mathcal{A} \otimes \mathbb{K}) : \ m \text{ is norm-smooth w.r.t. } \theta,\ a \in \mathcal{A}_{\mathcal{T},\theta}\}.\end{aligned}$$

This algebra is a Fréchet $*$-algebra w.r.t. the family of seminorms

$$\left\|(m, m+a)\right\|_{\mathscr{P}_j} = \left\|\nabla^j m\right\| + \left\|\nabla^j a\right\|_{\mathcal{T}},$$

for all multi-indices j, that is dense and spectral invariant in its C^*-completion.

Proof. The only point in question is the spectral invariance. We note that for $x_i = (m_i, m_i + a_i)$ one can write

$$\begin{aligned}\|x_1 \dots x_n\|_{\mathscr{P}_j} &= \left\|\nabla^j m_1 \dots m_n\right\| + \left\|\nabla^j((m_1 + a_1)\dots(m_n + a_n) - m_1 \dots m_n)\right\|_{\mathcal{T}}\\ &= \left\|\nabla^j m_1 \dots m_n\right\| + \left\|\nabla^j \sum_{\substack{k_1,\dots,k_n \in \{0,1\}\\ \sum k_i > 0}} y_{1,k_1} \dots y_{n,k_n}\right\|_{\mathcal{T}}\end{aligned}$$

with $y_i = (m_i, a_i)$, i.e. $y_{i,0} = m_i$, $y_{i,1} = a_i$. Denote by $\|\cdot\|_0$ the operator norm and $\|\cdot\|_1 = \|\cdot\|_{\mathcal{T}}$ then

$$\left\|\nabla^j \Big(\sum_{\substack{k_1,\dots,k_n \in \{0,1\}\\ \sum k_i > 0}} y_{1,k_1} \dots y_{n,k_n} \Big)\right\|_{\mathcal{T}} \leq \sum_{j_1 + \dots + j_n = j} \left\|\nabla^{j_1} y_{1,k_1}\right\|_{k_1} \dots \left\|\nabla^{j_n} y_{n,k_n}\right\|_{k_n}$$

since at least one k_i is equal to 1. Note that $\left\|\nabla^{j_i} y_{ik_i}\right\|_{k_1} \leq \|x_i\|_{\mathscr{P}_{j_1}}$ and thus

$$\|x_1 ... x_n\|_{\mathscr{P}_j} \leq 2 \sum_{j_1+...+j_n=j} \|x_1\|_{\mathscr{P}_{j_1}} \dots \|x_n\|_{\mathscr{P}_{j_n}}.$$

The increasing norms $\|x\|_{\widetilde{\mathscr{P}_m}} := \sum_{|j|\leq m} \|x\|_{\mathscr{P}_j}$ are therefore seen to satisfy the strong spectral invariance of Theorem 1.6.2. □

The operator-norm completion of $\mathscr{P}(\mathcal{A}, \mathcal{T}, \theta)$ has the form $\mathbb{P}(\mathcal{B}, \mathcal{A} \otimes \mathbb{K})$ with $\mathcal{B}$ the norm-completion of the algebra of smooth elements in $M^s(\mathcal{A})$, thus

$$\mathcal{B} = \{m \in M^s(\mathcal{A}) : t \in \mathbb{R}^d \mapsto \theta_t(m) \text{ is norm-continuous}\}.$$

It is not difficult to see that $K_i(\mathcal{B}) = 0$, indeed, the Eilenberg swindle argument used to prove that $K_i(M^s(\mathcal{A})) = 0$ (e.g. [20, Proposition 12.1.1]) generalizes verbatim, since all necessary elements are automatically also contained in $\mathcal{B}$. Thus $K_i(\mathbb{P}(\mathcal{B}, \mathcal{A} \otimes \mathbb{K})) \simeq K_i(\mathcal{A})$.

Definition 3.4.5 *On $\mathscr{P}(\mathcal{A}, \mathcal{T}, \theta)$ define the derivations*

$$\nabla_j(m, m + a) = (\nabla_j m, \nabla_j(m + a))$$

and the trace $\mathcal{T}^{\text{diff}}(m, m + a) := \mathcal{T}(a)$. Then the Chern cocycles

$$Ch^{\text{diff}}_{\mathcal{T},\theta}(x_0, \dots, x_n) \;=\; c_n \sum_{\rho \in S_n} (-1)^\rho \, \mathcal{T}^{\text{diff}}\big(x_0 \nabla_{\rho(1)} x_1 \cdots \nabla_{\rho(n)} a_n\big)$$

are cyclic n-cocycles on $\mathscr{P}(\mathcal{A}, \mathcal{T}, \theta)^{n+1}$.

To see that they are well-defined and cyclic cocycles is as easy as for the usual smooth subalgebras, since $\mathcal{T}^{\text{diff}}$ is a trace which satisfies the partial integration identity $\mathcal{T}^{\text{diff}}(x\nabla y) = -\mathcal{T}^{\text{diff}}((\nabla x)y)$.

Corollary 3.4.6 *Let n be odd and $[u_+]_1^M - [u_-]_1^M \in K_1(\mathcal{A})$ be represented by a couple $(u_+, u_-) \in \mathscr{P}(\mathcal{A}, \mathcal{T}, \theta)^{n+1}$, then*

$$\begin{aligned} &\langle Ch_{\mathcal{T},\theta}, [u_+]_1^M - [u_-]_1^M \rangle = \langle Ch^{\text{diff}}_{\mathcal{T},\theta}, [(u_+, u_-)]_1 \rangle \\ &= c_n \sum_{\rho \in S_n} (-1)^\rho \, \mathcal{T}^{\text{diff}}\big(u_+^* \nabla_{\rho(1)} u_+ \cdots \nabla_{\rho(n)} u_+^* - u_-^* \nabla_{\rho(1)} u_- \cdots \nabla_{\rho(n)} u_-^*\big). \end{aligned}$$

Let n be even and $[e_+]_0^M - [e_-]_0^M \in K_0(\mathcal{A})$ be represented by a couple $(e_+, e_-) \in \mathscr{P}(\mathcal{A}, \mathcal{T}, \theta)^{n+1}$, then

$$\begin{aligned}
&\langle Ch_{\mathcal{T},\theta}, [e_+]_0^M - [e_-]_0^M \rangle = \langle Ch^{\mathrm{diff}}_{\mathcal{T},\theta}, [(e_+, e_-)]_0 \rangle \\
&= c_n \sum_{\rho \in S_n} (-1)^\rho \, \mathcal{T}^{\mathrm{diff}}\big(e_+ \nabla_{\rho(1)} e_+ \cdots \nabla_{\rho(n)} e_+ - e_- \nabla_{\rho(1)} e_- \cdots \nabla_{\rho(n)} e_-\big).
\end{aligned}$$

Proof. The two pairings obviously coincide for a K_1-class represented by a unitary

$$(s(u), s(u) + (u - s(u)))$$

with $s(u)$ a scalar matrix, respectively a K_0-class $(s(e), s(e) + (e - s(e)))$. As argued before Definition 3.4.5, one has $K_i(\mathbb{P}(\mathcal{B}, \mathcal{A} \otimes \mathbb{K})) \simeq K_i(\mathcal{A})$ and thus every class has a representative of this form with $a \in \mathcal{A}_{\mathcal{T},\theta}$. This fixes all possible values of the pairing. □

Let us finally note that the domain of the difference cocycle as defined above is not natural in some cases. In applications one may want to use L^p-spaces different from L^1, e.g. to handle decay at infinity properly. In this smooth setting here one can always choose a regular enough representative and then relative it to various regularizations via continuity of the Chern cocycles w.r.t. L^p-norms.

4 Invariants of solid state systems

The object of study in this work are quantum-mechanical solid state systems in the free-fermion approximation, where an effective one-particle picture is used. We follow the C^*-algebraic approach which goes back to [15], thus a system is described by a Hamiltonian that is affiliated to a C^*-algebra. This makes it possible to classify topological invariants that are stable under carefully restricted homotopies of the Hamiltonian by labels in the K-theory of said algebra. There will also be situations where C^*-algebras will not be enough and one must pass to von Neumann algebra, e.g. if one needs to work with exact spectral projections. Then K-theory alone will not be helpful anymore, but the operator-algebraic approach still provides an access to index-theoretic methods.

The basic building block of observable algebras for a such a system will typically be a tracial dynamical system $(\mathcal{A}, \theta, \mathcal{T})$ consisting of a separable C^*-algebra $\mathcal{A}$ carrying a $\mathbb{R}^d$-action θ and a densely defined lower semicontinuous trace $\mathcal{T}$. In the semi-cyclic GNS representation induced by $\mathcal{T}$ the action is generated by d commuting self-adjoint generators $(X_1, ..., X_d)$ which one may think of as abstract position operators. The number of commuting generators is only a lower bound, which is not directly related to the physical extent of the system. For example, the edge of a d-dimensional system might be considered to be $(d-1)$-dimensional while one has in fact d independent position operators.

The importance of tracial dynamical systems as observable algebras is that in the analysis of topological quantum systems one encounters many different algebras which one can roughly split into algebras of more or less directly measurable observables and multiplier algebras, that are only observed through theirs action on observables. Using K-theory we can associate stable topological invariants to any projection or unitary in any C^*-algebra. Depending on the choice of algebra those K-groups will often be too small since they allow too many homotopies or too large in the sense that they have many classes that are distinguished only by invariants that are not physically interesting or depend subtly on the choice of algebra. For an observable algebra based on a tracial dynamical system $(\mathcal{A}, \theta, \mathcal{T})$, on the other hand, one can always compute at least a subset of the topological invariants canonically via the Chern number cocycles associated to n-parameter subgroups of θ and also associate index theorems to them which assure their stability under certain ”rough” perturbations. Moreover, the trace also has an important role in justifying at least some of the Chern numbers as transport coefficients since that is based on linear response theory (a similar algebraic point of view is taken by [44]). Let us also emphasize that in this work we are only concerned with the complex K-theory and even there mostly with the smaller part that can be labeled by Chern cocycles. Clearly, this sector of the theory of topological invariants is the most accessible and probably also the most stable one, especially since one has well-controlled computable numerical expressions for all invariants. Other invariants which require additional fundamental or spatial symmetries for stabilization or are $\mathbb{Z}_2$-valued will likely be much more susceptible to disorder. One can make meaningful statements, e.g. about the stability of $\mathbb{Z}_2$-invariants of mobility-gapped topological insulators (see e.g. the recent work [22]) but they require a considerably different approach and some basic questions are still unresolved.

4.1 Multipliers and unbounded Hamiltonians

The time-evolution of a system is generated by a self-adjoint operator H, called the Hamiltonian. For our purposes it is not sufficient that the group of automorphisms generated by H defines a strongly continuous action on some C^*-algebra $\mathcal{A}$, we more strongly require that H is a (possibly unbounded) multiplier affiliated to $\mathcal{A}$ (since we are interested in classifying its spectral projections). In the special case that $\mathcal{A}$ is unital, the Hamiltonian is always bounded and an actual element of $\mathcal{A}$ itself.

Definition 4.1.1 *Let $\mathcal{A}$ be a C^*-algebra. An $\mathcal{A}$-multiplier T shall be an operator affiliated to $M_N(\mathcal{A})$ in the sense of [132], i.e. a regular adjointable operator on the Hilbert module $\mathcal{A} \otimes \mathbb{C}^N$.*

A self-adjoint $\mathcal{A}$-multiplier is always a regular self-adjoint operator on the Hilbert module $\mathcal{A} \otimes \mathbb{C}^N$.

Usually it is preferred to deal with operators on a Hilbert space since the domains of the unbounded operators are more accessible. There is an equivalent characterization (see [132, Example 3,4]) for adapted to that situation:

Proposition 4.1.2 *If $\mathcal{A}$ acts faithfully and non-degenerately on a Hilbert space $\mathcal{H}$ then the $\mathcal{A}$-multipliers are in one-to-one correspondence with the closed operators T on $\mathcal{H}^N$ such that*

(i) $T^*(1+T^*T)^{-\frac{1}{2}} \in M(M_N(\mathcal{A}))$.

(ii) $(1+T^*T)^{-\frac{1}{2}}M_N(\mathcal{A}) \subset M_N(\mathcal{A})$ *is norm-dense.*

For further use we also want to highlight some special multipliers.

Definition 4.1.3 *An $\mathcal{A}$-multiplier T is called resolvent-affiliated if*

$$(T+z)^{-1} \in M_N(\mathcal{A})$$

for all $z \in \mathbb{C}$ in the resolvent set of T.

An $\mathcal{A}$-multiplier T is called strongly affiliated if

$$F(T) \in M_N(\mathcal{A}^\sim),$$

where $F(T) = T^*(1+T^*T)^{-\frac{1}{2}}$ is the bounded transform.

Depending on the physical situation to be modeled resolvent-affiliation of an unbounded Hamiltonian may be a fairly common property. If it does hold then one has very good stability under perturbations since additive perturbations in $M(\mathcal{A})$ will then lead to perturbations of the bounded transform in $\mathcal{A}$. Strong affiliation is also a good property to have, since its presence makes it easier to use T in the construction of lifts and preimages necessary for K-theoretic boundary maps, as we will see in Chapter 5.

In the following we will usually just write $\mathcal{A}$ instead of $M_N(\mathcal{A})$ and similarly for derived spaces $L^1(\mathcal{A})$ instead of $L^1(M_N(\mathcal{A}))$ since the dimension of the matrix fiber is usually obvious from context or irrelevant entirely (since one can often simply replace $\mathcal{A}$ with $M_N(\mathcal{A})$). Any bounded $\mathcal{A}$-multiplier is canonically included into the stable multiplier algebra $M^s(\mathcal{A})$ but the latter only features at intermediate steps since $\mathcal{A}$-multiplier are for us always finite size matrices (which has slight technical advantages).

A Hamiltonian will in this chapter always be a self-adjoint multiplier of a C^*-algebra $\mathcal{A}$. If $\mathcal{A}$ carries a $\mathbb{R}^d$-action θ then we say H is θ-smooth if H is strictly smooth in the sense of Definition 1.4.11 w.r.t. to the generators $X_1, \ldots, X_d$ of θ in some faithful covariant representation of $(\mathcal{A}, \mathbb{R}^d, \theta)$ (and only use consequences of that property which are representation-independent).

Concrete Hamiltonians are specified as (unbounded) operators in a Hilbert space representation but are then independent of it, since any representation of $\mathcal{A}$ comes with a canonical representation of its unbounded multipliers. A priori there is nothing wrong with an algebraic interpretation of quantum mechanics where $\mathcal{A}$ consists of observables which are related to measurement results using expectation values with respect to a trace $\mathcal{T}$, but one should always attempt to make explicit the connection of the algebraic approach to more functional analytic approaches to the analysis of (random) Schrödinger operators: In those concrete situations it is very common to have a preferred family $(\pi_\omega)_{\omega\in\Omega}$ of representations of $\mathcal{A}$ on some Hilbert spaces $(\mathcal{H}_\omega)_{\omega\in\Omega}$, where the elements of $\mathcal{H}_\omega$ have a direct physical interpretation e.g. as wave-functions of quasi-particles. The parameter space Ω can describe different realizations of a system, for example when the Hamiltonian is a random operator or depends on additional (deterministic) parameters. In that situation a single algebraic Hamiltonian H corresponds to a whole family of physical Hamiltonians $(\pi_\omega(H))_{\omega\in\Omega}$ and it can be difficult sometimes to translate algebraic statements into pointwise statements.

This problem becomes worse when one wants or needs to go beyond C^*-algebras, for example, to consider spectral projections or the polar decomposition of a Hamiltonian which does not have a spectral gap. We will then go over to the natural von Neumann algebra $L^\infty(\mathcal{A},\mathcal{T})$ constructed in Proposition 1.2.1, i.e. we take the completion $L^\infty(\mathcal{A},\mathcal{T}) = \pi_{\mathcal{T}}(\mathcal{A})''$ in the (unbounded) GNS-representation associated to $\mathcal{T}$. This has the advantage that θ extends to a weakly continuous action and $\mathcal{T}$ extends to a θ-invariant n.s.f. trace. Neither would be guaranteed in an arbitrary representation of $\mathcal{A}$. This puts us at the risk of losing contact with physical representations: It can happen that no π_ω extends to a normal representation of $L^\infty(\mathcal{A})$, or worse that $L^\infty(\mathcal{A})$ is an unphysical completion whose elements have little relation to e.g. spectral projections of actual physical observables $\pi_\omega(a)$. One way to alleviate this danger is to make certain that the GNS-representation is unitarily equivalent to the direct integral representation $\int_\Omega^\oplus \pi_\omega \mathrm{d}\mathbb{P}(\omega)$, where Ω now has a measurable structure with some measure $\mathbb{P}$. This is often just a matter of choosing Ω large enough initially. It is then guaranteed that $L^\infty(\mathcal{A})$ is decomposable, i.e. that any $a \in L^\infty(\mathcal{A})$ decomposes as a direct integral $\int_\Omega^\oplus a_\omega \mathrm{d}\mathbb{P}(\omega)$ with fibers that act on a physical representation of $\mathcal{H}_\omega$. This is enough to translate algebraic statements about an element $a \in L^\infty(\mathcal{A})$ into statements about physical observables a_ω that must hold on sets of positive measure w.r.t. $\mathbb{P}$, respectively must hold almost surely if $\mathbb{P}$ is ergodic in some sense. To consider semifinite index theorems for Chern numbers we must further go over to the auxiliary von Neumann algebra $L^\infty(\mathcal{A} \rtimes_\theta \mathbb{R}^n)$. If $L^\infty(\mathcal{A})$ is decomposable into a direct integral and the fibers π_ω are θ-covariant representations then $L^\infty(\mathcal{A} \rtimes_\theta \mathbb{R}^n)$ is again decomposable as a consequence of the spatial representation constructed in

Proposition 2.1.1. Thus the elements of $L^\infty(\mathcal{A} \rtimes_\theta \mathbb{R}^n)$ are again direct integrals with fibers acting on the physical Hilbert spaces.

4.2 Examples

The best-understood observable algebras take the form of (twisted) crossed products $C(\Omega) \rtimes G$ with $G = \mathbb{Z}^d$ or $G = \mathbb{R}^d$ depending on whether one considers a model on discrete or a continuous space and Ω describes an additional parameter space, e.g. a configuration space for random disorder. The crossed products here can be seen as special cases of groupoid C^*-algebras which are more flexible and can among other things also describe observable algebras where the real-space models live on uniformly discrete aperiodic sets (Delone sets) or quasicrystals (see e.g. [71, 17, 28, 27]). Yet, as is the line of argument in this chapter, the underlying structure as a crossed product or groupoid algebra does not really matter, it is only important for us as far as it is used to construct an action θ and a trace $\mathcal{T}$.

We describe in more detail two examples for observable algebras, the disordered non-commutative torus, which is used to handle random operators on $\ell^2(\mathbb{Z}^d)$ covariant under magnetic translations, as well as an observable algebra for continuous-space models, where the Hamiltonians are differential operators. Those two cases together with derived algebras, such as suspensions, edge or halfspace algebras already cover almost all of the expected phenomenology of topological insulators.

4.2.1 Disordered non-commutative torus

The disordered non-commutative torus is by now rather well-understood; we recall only as much detail as necessary and refer to [99, 103, 111]. It is most conveniently described as a twisted crossed product:

Definition 4.2.1 *Let G be an (additively written) discrete abelian group, $\mathcal{B}$ a C^*-algebra acting on a Hilbert space $\mathcal{H}$. Further let $\beta : G \times \mathcal{B} \to \mathcal{B}$ be a continuous G-action and $\rho : G \times G \to S^1$ a group 2-cocycle.*

The twisted crossed product $\mathcal{B} \rtimes_{\beta,\rho} G$ is the universal C^*-algebra generated by $\mathcal{B}$ and a set of unitary generators $(u^s)_{s\in G}$ satisfying

$$u^s u^t = \rho(s,t) u^{t+s}, \quad b u^s = u^s \beta_s(b)\,, \qquad s,t \in G, b \in \mathcal{B}\,. \tag{4.2.1}$$

If $\mathbf{B} \in M_d(\mathbb{C})$ is an anti-symmetric matrix then define the twist

$$\rho_{\mathbf{B}} : \mathbb{Z}^d \times \mathbb{Z}^d \to S^1, \quad (x, y) \mapsto e^{\imath \langle x, \mathbf{B}y \rangle}$$

and define the non-commutative torus

$$C(\mathbb{T}^d_{\mathbf{B}}) := \mathbb{C} \rtimes_{\mathrm{id}, \rho_{\mathbf{B}}} \mathbb{Z}^d$$

as a twisted crossed product with trivial action but magnetic twist. The notation is chosen on account of the natural Fourier isomorphism $C(\mathbb{T}^d_0) \simeq C(\mathbb{T}^d)$ for vanishing twist (hence $C(\mathbb{T}^d_{\mathbf{B}})$ is rather what the algebra of continuous functions on a non-commutative torus should be). This algebra, in a natural representation on $\ell^2(\mathbb{Z}^d)$ describes tight-binding models on the lattice $\mathbb{Z}^d$ which are covariant under magnetic translations induced by the magnetic field $\mathbf{B}$. They are still as translationally invariant as possible in the presence of a magnetic field.

Definition 4.2.2 *Let G be an abelian group. A measurable ergodic dynamical system $(\Omega, T, G, \mathbb{P})$ consists of a probability space $(\Omega, \mathbb{P})$ with Ω a compact metrizable Hausdorff space and $\mathbb{P}$ a regular Borel measure with full topological support. Let Ω be equipped with a continuous action $T : G \times \Omega \to \Omega$ under which $\mathbb{P}$ is invariant. The action T shall be ergodic, i.e. any measurable set $A \subset \Omega$ that is $\mathbb{Z}^d$-invariant up to sets of measure zero must have $\mathbb{P}(A) = 1$ or $\mathbb{P}(A) = 0$.*

The action is denoted by $T_x(\omega)$ for $x \in \mathbb{Z}^d$ and induces an action T^* on $C(\Omega)$ by $f \mapsto f \circ T_x$. Hence $C(\Omega)$ is a separable C^*-algebra on which integration w.r.t. $\mathbb{P}$ defines a continuous finite faithful trace that is invariant under T^*. The disordered non-commutative torus is then the crossed product

$$C(\mathbb{T}^d_{\mathbf{B},\Omega}) := C(\Omega) \rtimes_{T^*, \mathbf{B}} \mathbb{Z}^d.$$

Each elements of $C(\mathbb{T}^d_{\mathbf{B},\Omega})$ has a formal Fourier series representation

$$a = \sum_{x \in \mathbb{Z}^d} \psi(a)_x u^x$$

with continuous coefficient maps $\psi : C(\mathbb{T}^d_{\mathbf{B},\Omega}) \times \mathbb{Z}^d \to C(\Omega)$. There is a strongly continuous action

$$\theta : C(\mathbb{T}^d_{\mathbf{B},\Omega}) \times \mathbb{T}^d \to C(\mathbb{T}^d_{\mathbf{B},\Omega}), \quad \theta_t(a) = \sum_{x \in \mathbb{Z}^d} e^{2\pi \imath t \cdot x} \psi(a)_x u^x.$$

The correspondence with tight-binding models is provided by a family representations on the Hilbert space $\ell^2(\mathbb{Z}^d)N$ where u_j is represented by a magnetic translation and $C(\Omega)$ acting as multiplication operators [15, 103]:

Proposition 4.2.3 *A family $(\pi_\omega)_{\omega\in\Omega}$ of $*$-representations of $\mathcal{A}_d$ on $\ell^2(\mathbb{Z}^d)$ is on the generators given by*

$$\pi_\omega(u^{e_j}) \;=\; e^{\imath\langle e_j|\mathbf{B}_+|X\rangle} S^{e_j}\,, \qquad \pi_\omega(f) \;=\; \sum_{x\in\mathbb{Z}^d} f(T_x\omega)|x\rangle\langle x|\,,$$

where S^y denotes the shift operator acting by $S^y|x\rangle = |x+y\rangle$, $f \in C(\Omega)$ and $X = (X_1, ..., X_d)^T$ is a vector containing the unbounded position operators $\hat{X}_j|x\rangle = x_j|x\rangle$ on $\ell^2(\mathbb{Z}^d)$. The representations are non-degenerate and faithful for $\mathbb{P}$-almost all $\omega \in \Omega$.

An element a of $C(\mathbb{T}^d_{\mathbf{B},\Omega})$ therefore corresponds to a random family of operators $a = (\pi_\omega(a))_{\omega\in\Omega}$ where the dependence on ω is weak-operator-continuous. The best-known example for a Hamiltonian of that form is the magnetic Anderson model

$$\pi_\omega(h) = \sum_{|x|=1} \pi_\omega(u^x) + \omega_x|x\rangle\langle x|$$

with i.i.d. random variables $(\omega_x)_{x\in\mathbb{Z}^d} \in \Omega := [-1,1]^{\times\mathbb{Z}^d}$ forming a random on-site potential. In that representation the dual action is also generated by exponentiation of the commuting position operators $\hat{X}_i$. An element a is smooth w.r.t. θ if and only if the matrix elements $\|\psi(a)_x\| \sim |\langle x|\pi_\omega(a)|0\rangle|$ decay faster than any inverse polynomial in $\langle x\rangle$ corresponding to operators that have matrix elements with rapid off-diagonal decay in the standard basis.

Being a crossed product algebra there is a canonical trace induced by the trace $\mathbb{E} : C(\Omega) \to \mathbb{C}$, namely

$$\mathcal{T} : C(\mathbb{T}^d_{\mathbf{B},\Omega}) \to \mathbb{C}, \qquad \mathcal{T}(a) = \mathbb{E}(\psi_0(a)).$$

It is a finite continuous trace which is invariant under θ. Thus $C(\mathbb{T}^d_{\mathbf{B},\Omega})$ forms a tracial dynamical system with θ and $\mathcal{T}$. By virtue of ergodicity the trace $\mathcal{T}$ can be evaluated almost surely on fixed random configurations by a trace per unit volume:

$$\mathcal{T}(a) \overset{\text{a.s.}}{=} \frac{1}{(2L+1)^d} \sum_{x\in[-L,L]^d} \langle x|\pi_\omega(a)|x\rangle.$$

Let us also discuss the generated von Neumann algebra in the GNS-representation w.r.t. $\mathcal{T}$, which we denote by $L^\infty(\mathbb{T}^d_{\mathbf{B},\Omega})$. By construction of the dual trace the GNS-Hilbert space is canonically isomorphic to $L^2(\Omega \times \mathbb{Z}^d) \simeq L^2(\mathbb{T}^d_{\mathbf{B},\Omega})$ and since the L^2-norm is given by

$$\mathcal{T}(\sum_{x,y\in\mathbb{Z}^d} \psi_x(a^*)\psi_y(a)) = \sum_{x\in\mathbb{Z}^d} \mathbb{E}(\left|\psi_x(a)\right|^2) = \sum_{x\in\mathbb{Z}^d} \mathbb{E}\left|\langle x|\pi_\cdot(a)|0\rangle\right|^2 \tag{4.2.2}$$

it is not difficult to see that the GNS-representation unitarily equivalent to $\int_\Omega^\oplus \pi_\omega \mathrm{d}\mathbb{P}(\omega)$ via identification of $1_\Omega \otimes |0\rangle \in L^2(\Omega \times \mathbb{Z}^d)$ with the cyclic vector $\mathbb{1} \in L^2(\mathbb{T}^d_{\mathbf{B},\Omega})$. Thus we are in the situation described above and elements of $L^\infty(\mathbb{T}^d_{\mathbf{B},\Omega})$ still correspond to measurable families of random operators acting on the physical representation $\ell^2(\mathbb{Z}^d)$.

If Ω is $\mathbb{Z}^d$-equivariantly contractible to a point then $K_i(\mathbb{T}^d_{\mathbf{B},\Omega}) = K_i(\mathbb{T}^d_{\mathbf{B}})$ and the latter is computable via iteration of the Pimsner-Voiculescu exact sequence, one obtains

$$K_i(\mathbb{T}^d_{\mathbf{B}}) \simeq 2^{d-1},$$

for $i = 0, 1$ (but K_0 and K_1 are not canonically isomorphic to each other). We can associate to each projection 2^{d-1} even Chern numbers corresponding to a subset of $\{X_1, ..., X_d\}$ with an even number of elements and to a unitary 2^{d-1} odd Chern numbers for subsets of the generators with an odd number of elements. The values of those Chern cocycles are a complete list of invariants, i.e. they are in one-to-one correspondence with the K-theory class [103].

4.2.2 Continuous models

Many (effective) models in solid state physics are described by an unbounded Hamiltonian which acts on $L^2(\mathbb{R}^d)$ and which is of the form

$$H = D + V \tag{4.2.3}$$

with some differential operator D and a potential V which depends on space and may be (matrix-valued) constant, periodic, quasi-periodic or random. We call such models continuum models to distinguish them from tight-binding models which act on a Hilbert space $\ell^2(\Lambda)$ for a discrete set Λ. They are described by twisted crossed product algebras and for the brief presentation here we draw in particular from [76][29].

Due to its role in the Quantum Hall effect one of the most important examples is the magnetic Laplacian

$$\nabla_A^2 = -(\nabla + \iota A)^*(\nabla + \iota A)$$

for a covector field A that is the vector potential of a magnetic field. We assume the magnetic field $\mathbf{B}$ is constant in space such that one can encode it as above into an anti-symmetric matrix. In the symmetric gauge one then has

$$A_i(x) = -\frac{1}{2}\sum_{k=1}^{d} B_{ik}x_k.$$

The covariant derivatives $\nabla_A = \nabla + \iota A$ generate a projective representation of $\mathbb{R}^d$ with

$$e^{\nabla_A \cdot t} e^{\nabla_A \cdot s} = e^{\iota \langle t, \mathbf{B}s\rangle} e^{\nabla_A \cdot s} e^{\nabla_A \cdot t}.$$

The magnetic Laplacian is the Hamiltonian for a free non-relativistic particle in a constant magnetic field. Being unbounded this Hamiltonian cannot be an element of an observable algebra, however, its resolvents belong the twisted crossed product

$$C(\mathbb{R}^d_{\mathbf{B}}) := \mathbb{C} \rtimes_{\rho_{\mathbf{B}}} \mathbb{R}^d = C^*\left\{\int_{\mathbb{R}^d} f(t) e^{\nabla_A \cdot t} \mathrm{d}t,\ f \in C_c(\mathbb{R}^d)\right\}$$

with the twist as above determined by an anti-symmetric matrix $\mathbf{B}$, except now for an $\mathbb{R}^d$-action. For $d = 2$ this algebra is sometimes called the non-commutative Euclidean plane. There are many more Hamiltonians affiliated to this algebra, e.g. the magnetic Dirac-Hamiltonian $D = \sigma \cdot \nabla_A$ for σ a representation of the Clifford algebra $\mathbb{C}_d$ and others that we will study below in more detail.

We also want to allow periodic or random potentials. In that case our Hamiltonians will again be covariant families $(H_\omega)_{\omega\in\Omega}$ indexed by an ergodic dynamical system $(\Omega, T, \mathbb{R}^d, \mathbb{P})$. The admissibile Hamiltonians will then be of the form

$$H_\omega = D + \pi_\omega(V)$$

where $V \in C(\Omega)$ is represented on $L^2(\mathbb{R}^d)$ as the multiplication operator by a uniformly continuous function

$$(\pi_\omega(V)\psi)(x) = V(T_{-x}\omega)\psi(x), \quad \psi \in L^2(\mathbb{R}^d).$$

This representation is covariant with respect to the magnetic translations

$$e^{\imath\nabla_A\cdot x}\pi_\omega(V)e^{-\imath\nabla_A\cdot x}=\pi_{T_x\omega}(V)$$

from which one sees that π_ω integrates to a representation of the crossed product

$$C(\mathbb{R}^d_{\mathbf{B},\Omega}):=C(\Omega)\rtimes_{T^*,\rho_{\mathbf{B}}}\mathbb{R}^d.$$

This algebra has a $\mathbb{R}^d$-action θ dual to $\rho_{\mathbf{B}}$ with leaves the natural trace $\mathcal{T}$ invariant (the dual trace induced by the expectation value $\mathbb{E}:C(\Omega)\to\mathbb{C}$ given by integration over a probability measure $\mathbb{P}$).

If $\Omega=\{*\}$ is only one point and the magnetic field vanishes then $C(\mathbb{R}^d_{0,*})=C^*(\mathbb{R}^d)\simeq C_0(\mathbb{R}^d)$ is the group-C^*-algebra of $\mathbb{R}^d$, i.e. the C^*-closure of the convolution algebra $L^1(\mathbb{R}^d)$ which is via Fourier transform isomorphic to $C_0(\mathbb{R}^d)$. In this case a differential operator D also corresponds via Fourier transform to a polynomial in d variables. If that polynomial has no flat directions, then the resolvents of D are C_0-functions and resolvent-affiliation holds. In this momentum-space representation the action θ is given by translation and the trace $\mathcal{T}$ is the integral over the Lebesgue measure.

It is more difficult than for the non-commutative torus to explicitly construct good spaces Ω which yield realistic random models. For example, if $\Omega=\mathbb{T}^d\times[-1,1]^{\times\mathbb{Z}^d}$ then one can describe a random Hamiltonian such as the Anderson model with i.i.d. random variables and a periodic potential [76]

$$H_\omega=-\nabla_A^2+v_0(\cdot-r)+\sum_{y\in\mathbb{Z}^d}\omega_x\varphi(\cdot-y-r)$$

where v_0 is a $\mathbb{Z}^d$-invariant function, $\varphi\in C_0([0,1]^d)$ and $\omega=(r,(\omega_x)_{x\in\mathbb{Z}^d})$. Here the action on Ω is given by

$$T_y(r,(\omega_x)_{x\in\mathbb{Z}^d})=(r+y\mod\mathbb{T}^d,(\omega_{x+[y]})_{x\in\mathbb{Z}^d})$$

where $[y]$ is the integer part of $y\in\mathbb{R}^d$. The lift of the above potential to Ω is then

$$V(r,\omega)=v_0(r)+\omega_0\varphi(r).$$

Dropping the random part one can of course also have purely periodic potentials with $\Omega=\mathbb{T}^d$.

To describe the trace in the magnetic case or with a non-trivial potential, one notes that any regular enough element of $C(\mathbb{R}^d_{\mathbf{B},\Omega})$ can be written as an integral operator on $L^2(\mathbb{R}^d)$ with smooth integral kernel denoted by

$$K_a(\omega, x, y) = \langle x|\pi_\omega(a)|y\rangle.$$

Then the trace is the expectation of the diagonal term $\mathcal{T}(a) = \mathbb{E}\langle 0|\pi(a)|0\rangle$ and the L^2-norm equal to

$$\mathcal{T}(a^*a) = \int_{\mathbb{R}^d} \mathbb{E}\left|\langle 0|\pi(a)|x\rangle\right|^2 \mathrm{d}x. \tag{4.2.4}$$

From this equation it is also easy to see that $\int_\Omega^\oplus \pi_\omega \mathrm{d}\mathbb{P}(\omega)$ is equivalent to the GNS-representation on $L^2(\mathbb{R}^d_{\mathbf{B},\Omega})$ by identifying the kernel $\langle 0|\pi_\omega(a)|x\rangle$ with an element of $L^2(\mathbb{R}^d)$. The GNS-representation is only semi-cyclic, i.e. there is no cyclic vector (though the distribution $1_\Omega \otimes \delta_0$ formally plays the same role). As in the discrete case, one can also write $\mathcal{T}$ as a trace per unit-volume that almost surely does not depend on the configuration ω [80].

We can now examine the smoothness of elements of $C(\mathbb{R}^d_{\mathbf{B},\Omega})$ and affiliated Hamiltonians in the covariant representation on $L^2(\Omega \times \mathbb{R}^d)$ where the action θ is generated by the position operators

$$(X_j\phi)(\omega, x) = x_j\phi(\omega, x), \qquad \forall \phi \in L^2(\Omega \times \mathbb{R}^d).$$

There is some potential for confusion since the partial derivatives w.r.t. X (denoted ∇ in the previous chapters) must be distinguished from the $\mathcal{A}$-multipliers. For that reason we will write the derivatives w.r.t. θ as commutators with X in the following.

It is easy to see that elements of $C(\mathbb{R}^d_{\mathbf{B},\Omega})$ with a rapidly decaying kernel function are smooth, since the position operators act via

$$\langle 0|[X_i, \pi(a)]|x\rangle = -x_i\langle 0|\pi(a)|x\rangle.$$

For unbounded Hamiltonians one needs to verify the conditions of Definition 1.4.11. For example, the Pauli-Hamiltonian

$$H = -(\nabla + \imath A)^*(\nabla + \imath A)$$

satisfies the required domain inclusions and with the derivatives

$$[X_j, H] = -2(\nabla_j + \imath A_j)$$

the product $[X_j, H](1 + H^2)^{-\frac{1}{4}}$ extends to a bounded operator (it is know that $[X_j, H](1 + H)^{-1}$ extends to a bounded operator [23] and $(1 + H^2)^{-\frac{1}{4}}(1 + H)^{\frac{1}{2}}$ is bounded and invertible). Hence Definition 1.4.11 is satisfied with $\eta = \frac{1}{4}$. For trivial magnetic fields the smoothness of a more general Hamiltonian

$$H = P(\nabla_1, ..., \nabla_d) + V$$

with P a matrix of multi-variable polynomials and V a bounded potential is usually easy to verify with η depending on the order of P.

Let us point out that since $C(\mathbb{R}^d_{\mathbf{B},\Omega})$ can be written as an iterated crossed product with d copies of $\mathbb{R}$ one always has (due to the Connes-Thom isomorphism)

$$K_i(C(\mathbb{R}^d_{\mathbf{B},\Omega})) \simeq K_{i+d \bmod 2}(C(\Omega))$$

hence the K-theory depends heavily on the allowed family of potentials. In the absence of potentials or if the disorder space Ω is $\mathbb{R}^d$-equivariantly contractible one has in particular

$$K_i(C(\mathbb{R}^d_{\mathbf{B},*})) \simeq \begin{cases} \mathbb{Z}, & \text{if } d = i \mod 2, \\ 0, & \text{otherwise}, \end{cases}$$

with the single generator being a projection or unitary with non-vanishing top Chern number. On the other hand for periodic potentials all one has

$$K_i(C(\mathbb{R}^d_{\mathbf{B},\Omega})) \simeq K_i(C(\mathbb{T}^d)) \simeq K_{1-i}(C(\mathbb{T}^d))$$

and the class of each projection or unitary is specified precisely by the collection of 2^{d-1} even or odd weak Chern numbers generated by independent subgroups of θ.

4.3 Gapped topological invariants

In this section we work with an observable algebra $\mathcal{A}$ based on a tracial dynamical system $(\mathcal{A}, \theta, \mathcal{T})$. We use the von Neumann algebra $L^\infty(\mathcal{A})$ as constructed in

Proposition 1.2.1 and the L^p-spaces $L^p(\mathcal{A})$ where both may also refer to their matrix-valued versions $M_N(L^p(\mathcal{A}))$ if the size of the matrices is obvious from context. The same convention is used for Sobolev spaces $W^1_p(\mathcal{A})$ since there is no ambiguity possible.

As stated above, the time evolution of a non-interacting fermionic quantum system described by observables in or affiliated to $M(\mathcal{A})$ is fixed by a bulk one-particle Hamiltonian H which is a self-adjoint $\mathcal{A}$-multiplier. If H is bounded from below the ground state of a system at 0 temperature is described by the Fermi projection

$$e_F \;=\; \chi(H \le E_F) \;\in\; L^\infty(\mathcal{A})\,,$$

with the Fermi level $E_F \in \mathbb{R}$ being a given real number. For unambiguity it will often be necessary to assume that E_F is not an eigenvalue of H, though $E_F \in \sigma(H)$ is permissible. If E_F is not in the spectrum of H, i.e. if there is an actual spectral gap then $e_F \in M(\mathcal{A})$ lies in the multiplier algebra.

One sometimes also considers Hamiltonians which are bounded neither from above nor below. Such models usually arise either from effective models using Dirac-Hamiltonians or from differential equations of classical field theories that can be cast into a Hamiltonian formulation, most prominently the Maxwell equations and the shallow-water equations. In that case one must be careful with the interpretation of the Fermi projection as a ground state but it is still well-defined as an element of the multiplier algebra at least.

In most previous works on topological insulators assumptions were made that implied that the Fermi projection is not just an $\mathcal{A}$-multiplier but an actual matrix over the unitization $\mathcal{A}^\sim$. Indeed this is always the case automatically for a unital observable algebra, since all multipliers are bounded. For unbounded H, however, this is a subtle condition which sometimes fails, e.g. for Dirac-type Hamiltonians. There is an obvious reason why this issue rarely came up in previous works: Schrödinger-type operators are resolvent-affiliated to $\mathcal{A}$ and bounded from below, therefore their spectral projections are always matrices over $\mathcal{A}^\sim$ (since one can write the bounded transform $F(H) = \mathbb{1} + g(H)$ for a C_0-function g). However, even for bounded Hamiltonians of the tight-binding type one may run into problems if the observable algebra is non-unital and thus the Fermi projection is only a multiplier. For an example of the latter type see Section 4.3.5 below.

A problem with a topological classification of such Fermi projections is that the multiplier algebra can be very large and therefore might have too small K-groups and also not admit interesting cyclic cocycles to use as numerical invariants. In the extreme case of a stable algebra $\mathcal{A}$ the K-groups of $M(\mathcal{A})$ of course vanish outright. Thus there might be no topological invariants that are preserved under unrestricted Riesz-continuous homotopies of unbounded Hamiltonians. However, arbitrary homotopies are also not physically relevant; it is often the case that we look at classes of Hamiltonians which are additive perturbations of a fixed kinetic term H_0. Here the background Hamiltonian H_0 fixes an asymptotic dispersion relation, e.g. it tells us if we are dealing with a Schrödinger-type quasi-particle, a Dirac-type quasi-particle or something else entirely. Changing the asymptotic dispersion relation would be a very disruptive change and indeed one should not expect that any topological invariants survive such a shift. A good class of perturbations that allows topological classification should instead consist of operators that are bounded relative to H_0.

Motivated from the above discussion formulate our theory in terms of a reference Hamiltonian:

Definition 4.3.1 *A Hamiltonian H is a self-adjoint $\mathcal{A}$-multiplier. We say that a Hamiltonian H_0 is a reference Hamiltonian for H w.r.t. a spectral interval Δ if $\sigma(H_0) \cap \Delta = \emptyset$ and $F(H) - F(H_0) \in \mathcal{A}$.*

The bulk topological invariants will always be defined in comparison to some reference Hamiltonian and therefore depend on it, however, sometimes (and if $\mathcal{A}$ is unital then always) one can take H_0 to be a constant matrix, which is then the natural preferred choice.

It will often be that case that $H = H_0 + V$ with $V \in M(\mathcal{A})$, but the perturbation could also be an unbounded symmetric $\mathcal{A}$-multiplier. In many cases there is a natural choice of H_0 as the topologically trivial kinetic part of the Hamiltonian which becomes gapped at the Fermi level through introduction of a large mass term. If both the Hamiltonian and the reference Hamiltonian have a common spectral gap then the distinction is of course arbitrary and their roles can be interchanged. Note, however, that the definition is not symmetric since we require the reference Hamiltonian to always have an actual spectral gap, while we can also handle Hamiltonians that have only a mobility gap or a pseudogap. For the purposes of index theory alone one can also weaken $F(H) - F(H_0) \in \mathcal{A}$ to also allow differences in a larger subalgebra of $L^\infty(\mathcal{A})$. Nevertheless, the condition is imposed here for simplicity and since it ensures that the gapped topological invariants in the spectrally gapped case are actual pairings with $K_i(\mathcal{A})$.

The strongest results can be obtained when the Hamiltonian also has a spectral gap:

Definition 4.3.2 (Bulk gap hypothesis (BGH)) *The BGH is satisfied for a self-adjoint Hamiltonian H if the Fermi level E_F is contained in a spectral gap of H and H_0, i.e. there is a compact interval Δ with $E_F \in \Delta$ and $\Delta \cap \sigma(H) \cap \sigma(H_0) = \emptyset$.*

For odd topological invariants it is well-understood that the Hamiltonian must have an additional symmetry, namely it must anti-commute with a self-adjoint unitary J, which we will in the remainder of this work for simplicity assume is just a scalar matrix $J = \mathbb{1}_N \oplus (-\mathbb{1}_N)$ of appropriate size.

Definition 4.3.3 (Chiral hypothesis (CH)) *The CH holds for H if H and H_0 are odd in the grading of $\mathcal{A}$, i.e. H and H_0 anti-commute with J.*

Physically a chiral symmetry can be present for different reasons, e.g. due to a sublattice symmetry. The CH implies symmetry of the spectrum $\sigma(H) = -\sigma(H)$ and therefore it often makes sense to fix the Fermi level $E_F = 0$ if a chiral symmetry is present. The importance of a chiral symmetry is that it allows to reduce out from the Fermi projection a unitary building block, called the Fermi unitary. We can now define the main algebraic objects of interest:

Definition 4.3.4 *Let H be a Hamiltonian for which $E_F \in \mathbb{R}$ is not an eigenvalue and H_0 a reference Hamiltonian for H with a spectral gap that contains E_F.*

Define the Fermi projections

$$e_F = \chi(H < E_F), \qquad e_0 = \chi(H_0 < E_F)$$

and if the CH is satisfied and $E_F = 0$ define the Fermi unitaries u_F, u_0 as the off-diagonal parts of

$$\operatorname{sgn}(H) = \begin{pmatrix} 0 & u_F^* \\ u_F & 0 \end{pmatrix}, \quad \operatorname{sgn}(H_0) = \begin{pmatrix} 0 & u_0^* \\ u_0 & 0 \end{pmatrix}$$

in a grading where $J = \operatorname{diag}(1, -1)$.

If the BGH also holds for a common gap $E_F \in \Delta$ of H and H_0 then we associate to the gap the class

$$[e_F]_0^M - [e_0]_0^M \in K_0(\mathcal{A})$$

and in the chirally symmetric case also

$$[u_F]_1^M - [u_0]_1^M = [u_F u_0^*]_1 \in K_1(\mathcal{A}).$$

Here we use the multiplier picture of K-theory as it is defined in Section 1.5. Note that here

$$\chi(H < \Delta) - \chi(H_0 < \Delta) \in \mathcal{A}$$

and $u_F - u_0 \in \mathcal{A}$ since the assumption $F(H) - F(H_0) \in \mathcal{A}$ implies that $f(F(H)) - f(F(H_0)) \in \mathcal{A}$ of all functions f which are bounded and continuous on $\sigma(F(H)) \cup \sigma(F(H_0))$.

Under the BGH (and CH) numerical topological invariants are well-defined as pairings between K-theory and cyclic cohomology

$$\langle \mathrm{Ch}_{\mathcal{T},\alpha}, [e_F]_0^M - [e_0]_0^M \rangle, \qquad \langle \mathrm{Ch}_{\mathcal{T},\alpha}, [u_F]_1^M - [u_0]_1^M \rangle$$

for α any restriction of θ to an n-parameter subgroup. The pairings do not require any smoothness and integrability assumptions on H, since one can always choose a regular enough representative of the respective K-theory class.

A related construction for the relative pairings was proposed in [13] for pairs of translation-invariant Hamiltonians affiliated to $C(\mathbb{R}^d_{0,*}) \simeq C_0(\mathbb{R}^d)$ whose Fermi projections $e_\pm \in M_N(C_b(\mathbb{R}^d))$ are asymptotically equal. They can then be combined into a class in $K_0(S^d)$ by gluing them together along the boundary at infinity and a numerical Chern number can be defined via the canonical Chern cocycle on the sphere. The approach here using stable multipliers generalizes much more easily to the non-commutative case and gives equivalent invariants. It was developed in the course of a parallel work [77]. The point of view that topological phases may only be defined relatively with respect to a reference state is particularly natural in van Daele K-theory and has also been emphasized in other recent works (e.g. [2, 24]).

For some technical points we need to track in which L^p-spaces the difference of Fermi projections lies. Since in the general case $e_F \in L^\infty(\mathcal{A})$ is the only inclusion into an L^p-space available, all regularity will need to come from the perturbation:

Definition 4.3.5 *We say that H is a p-smooth perturbation of the reference Hamiltonian H_0 if H and H_0 are strictly smooth in the sense of Definition 1.4.11 and $F(H) - F(H_0)$ is in $W_p^\infty(\mathcal{A})$.*

We say that $V = V^* \in M(\mathcal{A})$ is a p-smooth potential if V is strictly smooth and there is a fraction $0 < s < 1$ such that $V(1 + H^2)^{-\frac{s}{2}} \in L^p(\mathcal{A})$.

If $H = H_0 + V$ for a p-smooth potential then H is a p-smooth perturbation of H_0 by Proposition 1.4.19. In particular, if the resolvent of H is in $L^p(\mathcal{A})$ then any strictly smooth potential V is a p-smooth potential. Nevertheless, one can also have perturbations for which $H - H_0$ is unbounded.

Irrespective of further regularity assumption we can then control certain smooth functions of the Hamiltonian:

Proposition 4.3.6 *Let $\Theta \in C^\infty([-1, 1])$ be a smooth function. If H is a p-smooth perturbation of a reference Hamiltonian H_0 then*

$$\Theta(F(H)) - \Theta(F(H_0)) \in W^\infty_p(\mathcal{A}).$$

Proof. Follows immediately from the smooth functional calculus since

$$(F(H) - z)^{-1} - (F(H_0) - z)^{-1} = (F(H_0) - z)^{-1}(F(H) - F(H_0))(F(H) - z)^{-1}$$

and thus

$$\left\|\nabla^j((F(H) - z)^{-1} - (F(H_0) - z)^{-1}\right\|_p \leq c_j \sum_{m=1}^{|j|} |\Im m\, z|^{-1-m}$$

for constants c_j depending only on $\left\|\nabla^j(F(H) - F(H_0))\right\|_p$. □

Notably, if H and thus $F(H)$ has a spectral gap then the Fermi projection can be written as such a smooth function.

If H does not have a spectral gap in Δ we can still consider the Fermi projection as an object in the von Neumann algebra $L^\infty(\mathcal{A})$ and may sometimes still define the Chern numbers as topological invariants to serve as proxies for K-theory. Without a spectral gap there is in general no way to recover from the Fermi projection/unitary a class in $K_i(\mathcal{A})$ anymore, hence one needs to carefully ensure that the Fermi projection/unitary already lie in the domain of some or all Chern cocycles, i.e. Sobolev spaces of the correct order. By referring to the case $\mathcal{A} = C_0(\mathbb{R}^d)$ it is easy to see that a spectral projection for an arbitrary spectral interval will often have similar smoothness properties as a function with jump discontinuities (unless the endpoints of the interval lie in spectral gaps). Therefore we will eventually require additional properties for the spectrum around the Fermi level to avoid those pathologies: The two main settings of interest in this work are that of *pseudogap* where there are only very little states around the Fermi level such that singularities in the Fermi projection are dimensionally suppressed, the other is that of a *mobility gap*, i.e. we stipulate a spectral region which consists only of localized states and where the Fermi projection has rapid off-diagonal decay in an averaged sense.

A crucial point is that one needs to control the difference between an approximation and an actual spectral projection in terms of L^p- and Sobolev norms. One notion that will be important in both of the pseudogapped and mobility gapped setting is the density of states:

Definition 4.3.7 *Let H be a self-adjoint $\mathcal{A}$-multiplier. We say that the spectral interval $\Delta \subset \mathbb{R}$ is $\mathcal{T}$-finite if $\chi(H \in \Delta') \in L^1(\mathcal{A})$ for each compact interval $\Delta' \subset \Delta$. Then one has the integrated density of states (IDOS)-measure as the unique regular Borel measure such that*

$$\mathcal{T}(f(H)) = \int_{\mathbb{R}} f(H) \mathrm{d}\nu_{\Delta,H}, \qquad \forall f \in C_c(\Delta).$$

The DOS of H is γ-Hölder continuous at E_0 if there is an open interval I, $E_0 \in I$ and a constant C such that for all $\epsilon > 0$ with $[E_0 - \epsilon, E_0 + \epsilon] \subset I$

$$\nu_{\Delta,H}([E_0 - \epsilon, E_0 + \epsilon]) \;\leq\; C\,\epsilon^{\gamma}\,. \tag{4.3.1}$$

Often the density of states is the limit of the actual number of eigenvalues per unit volume in finite-volume approximations of H.

4.3.1 Mobility gaps

A spectral interval Δ of a random Schrödinger-type Hamiltonian $(H_\omega)_{\omega\in\Omega}$ is called a mobility gap if H_ω almost surely has only dense point spectrum Δ with exponentially localized eigenfunctions. States from the localized region do not contribute to the direct conductivity and it is understood that e.g. in the situation of the Quantum Hall effect mobility gaps support almost the same physical phenomenology as a true spectral gap [16], in fact may be required for a full description. On a technical level there are different overlapping notions that one can use for a mobility gap, but a central aspect is that the time evolution

$e^{-\imath Ht}\chi(H \in \Delta)$ should have matrix elements whose disorder-averaged norm decays exponentially with rates that are uniform in t. Indeed, one can usually establish the slightly stronger property that for Borel functions f the decay rate of the matrix elements of $P_x f(H)\chi(H \in \Delta)P_y$ does not depend on the smoothness of f. By matrix elements we mean as in Section 2.1 that one should sandwich functions bounded function of H with indicator functions of the position operators $P_x g(H) P_y$. This will be made more precise further below and it will also be show that this decay is already characterized completely by the Sobolev norms of $W_p^\infty(\mathcal{A})$ without a need to consider matrix elements. Thus we are lead to a similar definition as in [28]:

Definition 4.3.8 *We say that H has a mobility gap in the $\mathcal{T}$-finite spectral interval Δ if the density of states of H is γ-Hölder continuous in Δ for some $0 < \gamma \leq 1$ and for each $r \in [1, \infty)$ the map*

$$f \in B(\Delta) \mapsto f(H) \in W_r^\infty(\mathcal{A}) \tag{4.3.2}$$

is bounded where $B(\Delta)$ are the bounded Borel functions which vanish outside Δ. Here the left-hand side is supplied with the supremum norm and the right-hand side with the natural Fréchet topology.

It is important here that the norms of $W_r^\infty(\mathcal{A})$ for $r \neq \infty$ are defined in terms of a trace which includes some form of disorder average (see e.g. (4.2.2) or (4.2.4))and is therefore significantly weaker than decay in operator-norm. As we will show further below it is actually enough to establish (4.3.2) for any single $1 \leq r < \infty$ since one can interpolate using the finite density of states. In practice the formulations with $r = 1$ or $r = 2$ fixed are easiest to relate to other characterizations from the literature. In particular, the mobility gap condition here is easily implied by exponential decay of the disorder-averaged eigenfunction correlator as it is used in [6]. For random tight-binding models on $\ell^2(\mathbb{Z}^d)$ that condition reads

$$\sum_{x,y \in \mathbb{Z}^d} e^{\mu|x-y|} \int_\Omega \sup_{\substack{f \in B(\Delta) \\ \|f\|_\infty \leq 1}} \left\| \langle x | f(H_\omega) | y \rangle \right\| \, \mathrm{d}\mathbb{P}(\Omega) < \infty.$$

and implies that the random Hamiltonians H_ω have at most dense pure point spectrum in Δ with exponentially decaying eigenfunctions under mild additional assumptions on the spectral degeneracy (see e.g. [7]).

We included γ-Hölder continuity of the density of states into the definition of a mobility for technical convenience; for some results having a density of states measure without atoms in Δ is sufficient. The latter is necessary to exclude the possibility that H has an eigenvalue in Δ (note however that if H represents a random family of operators then this only excludes the possibility that any fixed energy is an eigenvalue with positive probability).

Proposition 4.3.9 *If H is a p-smooth perturbation of some H_0 with spectral gap in Δ and H has a mobility gap in Δ then*

$$e_F - e_0 \in W_p^\infty(\mathcal{A}).$$

Proof. Write $\chi(H < E_F) = \Theta(F(H)) + g(H)$ as the sum of a smooth function Θ whose derivative is compactly supported within Δ and a bounded Borel function g supported in Δ. Then obviously

$$e_F - e_0 = g(H) + \Theta(F(H)) - \Theta(F(H_0)) \in W_p^\infty(\mathcal{A})$$

by Proposition 4.3.6 and the definition of a mobility gap. □.

If one has a smooth family of perturbations which preserves the mobility gap then the Fermi projection is also continuous:

Proposition 4.3.10 *Let $t \in [0,1] \to H(t)$ be a path of the form $H_t = H_0 + V_t$ for H_0 a reference Hamiltonian with spectral gap Δ, with a norm-continuous family of p-smooth potentials $V \in M(\mathcal{A})$ which are such that there is a common mobility gap Δ in the sense that the maps of (4.3.2) are uniformly continuous w.r.t. t and the the density of state measures are uniformly γ-Hölder continuous in the sense that*

$$\nu_{\Delta,H(t)}([E_0, E_1]) \;\leq\; C\;|E_1 - E_0|^\gamma\;, \qquad \forall E_0, E_1 \in \Delta : E_0 < E_1,$$

with constant C independent of t.

Then the difference of Fermi projection $t \mapsto e_F(t) - e_0$ is continuous w.r.t. the topology of $W_p^\infty(\mathcal{A})$.

Proof. From Proposition 1.4.19 one concludes that $\Theta(F(H(t))) - \Theta(F(H_0))$ is continuous w.r.t. to the topology of $W_p^\infty(\mathcal{A})$ for any switch function Θ as in Proposition 4.3.6. Choose for each $\delta > 0$ a positive switch function $0 \leq g_\delta \leq 1$ of the form $g_\delta = \Theta_\delta \circ F$ such that g'_δ is supported in the interval $(E_F - \delta, E_F + \delta)$ within Δ. Due to the uniform Hölder continuity we can ensure that

$$\left\| g_\delta(H(t)) - e_F(t) \right\|_p \leq \left(2\nu_{\Delta,H(t)}([E_F - \delta, E_F + \delta])\right)^{\frac{1}{p}} \leq c\delta^{\frac{\gamma}{p}}$$

for a constant independent of t. By a $\frac{\epsilon}{3}$-argument $t \mapsto e_F(t) - e_0$ is therefore continuous in the norm of $L^p(\mathcal{A})$.

Let j be a multi-index and χ_δ the indicator function for the interval $E_F + (-\delta, \delta)$, thus

$$g_\delta(H(t)) - e_F(t) = \chi_\delta(H(t))(g_\delta(H(t)) - e_F(t))$$

which implies

$$\left\|\nabla^j (g_\delta(H(t)) - e_F(t))\right\|_p \leq \sum_{j_1+j_2=j} \left\|\nabla^{j_1}(g_\delta(H(t)) - e_F(t))\right\|_r \left\|\nabla^{j_2}\chi_\delta(H(t))\right\|_q$$

for any combination $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$ with $1 < r, q < \infty$. Since χ_δ is idempotent we have

$$\left\|\nabla^{j_2}\chi_\delta(H(t))\right\|_q = \left\|\nabla^{j_2}\left(\chi_\delta(H(t))^{|j_2|+1}\right)\right\|_q \leq C \left\|\chi_\delta(H)\right\|_{(|j_2|+1)q}$$

with a constant depending only on the norms $\left\|\nabla^k \chi_\delta(H)\right\|_{(|j_2|+1)q}$ for $k \leq j_2$ and those are bounded independent of δ by the mobility gap assumption. We conclude

$$\left\|\nabla^j (g_\delta(H(t)) - e_F(t))\right\|_p < C\delta^{\frac{\gamma}{(|j|+1)q}}$$

and thus

$$\begin{aligned}
\left\|\nabla^j (e_F(t) - e_0)\right\|_p &\leq \left\|\nabla^j (g_\delta(H(t)) - e_0)\right\|_p + \left\|\nabla^j (g_\delta(H(t) - e_F(t))\right\|_p \\
&\leq \left\|\nabla^j (g_\delta(H(t)) - g_\epsilon(H_0)\right\|_p + C\delta^{\frac{\gamma}{(|j|+1)q}}
\end{aligned}$$

implies continuity of $t \mapsto e_F(t) - e_0$ w.r.t. each seminorm of $W^\infty_p(\mathcal{A})$ via an $\frac{\epsilon}{3}$-argument since $g_\delta(H(t)) - g_\delta(H(0))$ is continuous for each δ. □

The remainder of this section is slightly outside the main line of development but important for Chapter 6 and to understand the relation with other localization criteria. The goal is to derive consequences from the mobility gap for (averaged) matrix elements. Define for $x \in \mathbb{Z}^d$ the projection onto a cube

$$P_x = \chi(X \in x + [0,1)^d) \;\; \in L^\infty(\mathcal{A}) \rtimes_\theta \mathbb{R}^d$$

for $X = (X_1, ..., X_d)$ the generators of the action (denoted D in Chapter 2). From Proposition 2.1.1 we know that if $L^\infty(\mathcal{A})$ acts on a Hilbert space $\mathcal{H}$ with θ the action of some commuting position operators $\hat{X}_1, ..., \hat{X}_d$ then the abstract matrix element $P_x a P_y \in L^\infty(\mathcal{A} \rtimes_\theta \mathbb{R}^d)$ is the direct integral of all translates

$$\chi(\hat{X} \in [r+x, r+x+1_d)^d) a \chi(\hat{X} \in [r+y, r+y+1_d)^d).$$

Therefore a decay estimate for the matrix elements in the algebra $L^\infty(\mathcal{A} \rtimes_\theta \mathbb{R}^d)$ is practically the same as a translation-invariant (averaged) decay estimate in a physical representation.

Using L^s-quasi-norms allows an operator-algebraic formulation of the so-called Aizenman-Molchanov fractional moments bound [5, 6]:

Definition 4.3.11 *We say that the self-adjoint multiplier H satisfies a fractional moments bound if for any smooth function φ supported in Δ, there is some fractional power $0 < s < 1$ such that*

$$\sup_{x,y \in \mathbb{Z}^d} \left\| P_x \varphi(H)(H+z)^{-1} P_y \right\|^s_{L^s(\mathcal{A} \rtimes_\theta \mathbb{R}^d)} \langle x - y \rangle^k \leq A_k < \infty$$

and

$$\sup_{x,y \in \mathbb{Z}^d} \left\| P_x \varphi(H)(F(H)+z)^{-1} P_y \right\|^s_{L^s(\mathcal{A} \rtimes_\theta \mathbb{R}^d)} \langle x - y \rangle^k \leq A_k < \infty.$$

This or a very similar version of a localization bound is widely used in the analysis of topological insulators (e.g. [4, 100, 103, 99, 29, 111] since it rather easily translates into decay estimates for the Fermi projection via contour integral representations. Unfortunately, it is not exactly clear how ubiquitous it is, since especially for continuous models one often uses methods for establishing spectral localization that do not directly imply a fractional moments bound, most importantly the multiscale method of [55]. On the other hand, the recent result [110] shows that already a deterministic analogue of Definition 4.3.8 implies a fractional moments bound for the energy-averaged resolvent. The goal for the remainder of the section is to show that the fractional moments bounds as above in fact can be derived from our notion of mobility gap due to the Hölder-continuity of the density of states. This will be instrumental in Section 6.1.

The following characterization will be useful:

Proposition 4.3.12 *Let $X_1, ..., X_d$ be the position operators affiliated to $L^\infty(\mathcal{A} \rtimes_\theta \mathbb{R}^d)$ and denote*

$$P_x = \chi(X \in x + [0,1)^d).$$

We call an element $a \in L^p(\mathcal{A}) \cap L^\infty(\mathcal{A})$ spatially p-smooth for $0 < p \leq \infty$ if

$$\sup_{x,y} \left\| P_x a P_y \right\|_{L^p(\mathcal{A} \rtimes_\theta \mathbb{R}^d)} \langle x - y \rangle^N < \infty \tag{4.3.3}$$

for each $N > 0$.

Any element of $L^\infty(\mathcal{A}) \cap W_p^\infty(\mathcal{A})$, $1 \leq p < \infty$, is spatially p-smooth and the converse holds for all integers $p \in \mathbb{N}$.

Proof. Assume that a is in $L^\infty(\mathcal{A}) \cap W_p^\infty(\mathcal{A})$, then by Proposition 1.4.5 and Proposition 2.1.5 there is for each k a constant such that

$$\left\| P_x a P_y \right\|_{L^p(\mathcal{A} \rtimes_\theta \mathbb{R}^d)} = \left\| P_x (\widehat{W_j} * a) P_y \right\|_{L^p(\mathcal{A} \rtimes_\theta \mathbb{R}^d)} \leq c_k 2^{-kj}$$

whenever $|x - y| > 2^{j+3}$. Since

$$\sup_{|x|>1} 2^{-k \log|x|} \langle x \rangle^N < \infty$$

for $k \geq N$ this implies (4.3.3).

On the other hand, if (4.3.3) is finite for each N and $p = m$ then we can write the L^m-norm using Corollary 2.1.4 as

$$\begin{aligned}
\|a\|_{L^m(\mathcal{A})}^m &= \hat{\mathcal{T}}_\xi (P_0 \left| a \right|^m P_0) \leq \|a^m P_0\|_1 \\
&\leq \sum_{x_1,\ldots,x_m \in \mathbb{Z}^d} \left\| P_{x_1} a P_{x_2} \ldots P_{x_m} a P_0 \right\|_1 \\
&= \sum_{x_1,\ldots,x_m \in \mathbb{Z}^d} \left\| P_{x_1} a P_{x_2} \right\|_m \left\| P_{x_2} a P_{x_3} \right\|_m \cdots \left\| P_{x_m} a P_0 \right\|_m \\
&= \sum_{y_1,\ldots,y_m \in \mathbb{Z}^d} \left\| P_{y_1} a P_0 \right\|_m \left\| P_{y_2} a P_0 \right\|_m \cdots \left\| P_{y_m} a P_0 \right\|_m \\
&= \left(\sum_{y \in \mathbb{Z}^d} \left\| P_{y_m} a P_0 \right\|_m \right)^m
\end{aligned}$$

where the second-to-last equality used that invariance of the trace under the dual action $\hat{\theta}$ implies the shift-invariance $\left\| P_x a P_y \right\|_m = \left\| P_{x-z} a P_{y-z} \right\|_m$.

Since P_x is θ-invariant we also have

$$\left\|P_x(\widehat{W_j} * a)P_y\right\|_p = \left\|\widehat{W_j} * (P_x a P_y)\right\|_{L^p(\mathcal{A})} \leq \left\|P_x a P_y\right\|_p$$

which shows that each piece in the dyadic decomposition is also spatially p-smooth with the same constants. Hence it is easy to see that

$$\left\|\widehat{W_j} * a\right\|_p \leq \sum_{\substack{y\in\mathbb{Z}^d \\ 2^{j-1}<|y|<2^{j+3}}} \left\|P_y a P_0\right\|_m$$

decays faster than 2^{-jk} for any k, which implies that a is in any Besov space $B^s_{p,\infty}(\mathcal{A})$ and thus smooth w.r.t. to the topology of $W^\infty_p(\mathcal{A})$ by Proposition 1.4.5.□

This characterization implies that indeed it is sufficient to require that (4.3.2) from Definition 4.3.8 only holds for any single $r \in [1, \infty)$, since

$$\left\|P_x f(H) P_y\right\|_{L^p(\mathcal{A}\rtimes\mathbb{R}^d)} \leq \left\|P_x f(H) P_y\right\|^{1-\theta}_{L^p(\mathcal{A}\rtimes\mathbb{R}^d)} \, \|f\|^\theta_\infty$$

by interpolation for any $p \in [r, \infty)$ and suitable $0 < \theta < 1$, likewise

$$\begin{aligned}\left\|P_x f(H) P_y\right\|_{L^p(\mathcal{A}\rtimes\mathbb{R}^d)} &\leq \left\|P_x f(H) P_y\right\|^{1-\theta}_{L^r(\mathcal{A}\rtimes\mathbb{R}^d)} \, \left\|f(H)\right\|^\theta_{L^\gamma(\mathcal{A}\rtimes\mathbb{R}^d)} \\ &\leq \left\|P_x f(H) P_y\right\|^{1-\theta}_{L^r(\mathcal{A}\rtimes\mathbb{R}^d)} \|f\|^\theta_\infty \, \nu_{\Delta,H}(\Delta)^{\theta/\gamma}\end{aligned}$$

for $p \in [1, r)$.

The smoothness alone already implies rapid decay of matrix elements in operator norm:

Proposition 4.3.13 *If H is θ-smooth then $(H + z)^{-1}$ and $(F(H) + z)^{-1}$ are ∞-smooth with*

$$\sup_{x,y\in\mathbb{Z}^d} \left\|P_x(H + z)^{-1}P_y\right\| \langle x - y\rangle^K \leq C_K \sum_{m=0}^{K+d} |\Im m\, z|^{-1} \left(1 + \frac{\langle \Re e\, z\rangle}{|\Im m\, z|}\right)^m$$

and

$$\sup_{x,y\in\mathbb{Z}^d} \left\|P_x(F(H) + z)^{-1}P_y\right\| \langle x - y\rangle^K \leq C_K \, |\Im m\, z|^{-K-d-1}.$$

For the off-diagonal elements the first bound can be strengthened to

$$\sup_{\|x-y\|_\infty>2} \left\|P_x(H+z)^{-1}P_y\right\| \langle x-y\rangle^K \leq C_K \sum_{m=1}^{K+d} |\Im m\, z|^{-1-m(1-2\eta)} \left(1+\frac{\langle \Re e\, z\rangle}{|\Im m\, z|}\right)^{m(1-2\eta)}$$

with $0<\eta<\frac{1}{2}$ *the fractional exponent from Definition 1.4.11.*

Proof. From the boundedness of seminorms of Proposition 4.3.12 one can read off

$$\left\|(H+z)^{-1}\right\|_{\mathscr{S}_{\infty,K}} \leq C_K \sum_{|j|\leq K} \left\|\nabla^j (H+z)^{-1}\right\|$$

and similarly for the resolvent of $F(H)$. For the refinement we notice that the off-diagonal components can be estimated in terms of $\widehat{W_j} * (H+z)^{-1}$ with $j>1$ alone, thus the 0-th order term drops out to give

$$\left\|P_x(H+z)^{-1}P_y\right\|_{\mathscr{S}_{\infty,K}} \langle x-y\rangle^K \leq C_K \sum_{1\leq|j|\leq K} \left\|\nabla^j (H+z)^{-1}\right\|.$$

The natural estimates for $\left\|\nabla^j(H+z)^{-1}\right\|$ in terms of $|||H|||_{j,\eta}$ then give the result. □

This result is an abstract version of a so-called Combes-Thomas estimate. It characterizes the expected behavior of the off-diagonal matrix elements, namely they decay in operator norm faster than any polynomial but with rates that blow up as one approaches the spectrum. The mobility gap in contrast asserts that in the L^p-norms for $1<p<\infty$ the norms are at worst proportional to the height of the resolvent $\sim |\Im m\, z|^{-1}$ but with otherwise uniform decay. We can control that better by taking into account that the resolvent becomes sharply peaked for $|\Im m\, z| \to 0$ and using Hölder-continuity of the density of states:

Proposition 4.3.14 *Assume that* H *is a* θ*-smooth multiplier with a mobility gap in* Δ *and has a* γ*-Hölder-continuous density of states in* Δ*. For any* $k\in\mathbb{N}$, $0<s<1$ *and* $0<\epsilon<1$ *there is a constant* $C_{k,s,\epsilon}$ *such that for any bounded Borel function* f *supported in an interval* $\Delta'\subset\Delta$ *one has*

$$\left\|P_x f(H) P_y\right\|_s \leq C_{k,s,\epsilon} \|f\|_{L^1(\Delta,\nu_{H,\Delta})}^{1-\epsilon} \|f\|_\infty^\epsilon |\Delta'|^{\gamma\frac{1-s}{s}(1-\epsilon)} \langle x-y\rangle^{-k}.$$

Proof. For any $0 < s < 1$ fix the unique $K \in \mathbb{N}$ for which $1 < \frac{Ks}{1-s} \leq 2$ and define the characteristic function $\chi = \chi(H \in \Delta')$ such that $f = f\chi = f\chi^K$. Then the Hölder inequality gives

$$\left\|P_x f(H) P_y\right\|_s^s \leq \sum_{\substack{z_1,\dots,z_{K+1}\in\mathbb{Z}^d \\ z_{K+1}=y}} \left\| P_x f(H) P_{z_1} \prod_{k=1}^{K} P_{z_k} \chi(H) P_{z_{k+1}} \right\|_s^s$$
$$\leq \sum_{\substack{z_1,\dots,z_{K+1}\in\mathbb{Z}^d \\ z_{K+1}=y}} \left\|P_x f(H) P_{z_1}\right\|_1^s \prod_{k=1}^{K} \left\|P_{z_k}\chi(H)P_{z_{k+1}}\right\|_{\frac{Ks}{1-s}}^s .$$

For arbitrary $0 < \epsilon < 1$, $m \in \mathbb{N}$ and $1 \leq p \leq 2$ we estimate

$$\left\|P_x g(H) P_{z_1}\right\|_p = \left\|P_x f(H) P_{z_1}\right\|_p^{1-\epsilon} \left\|P_x f(H) P_{z_1}\right\|_p^{\epsilon}$$
$$\leq \|g\|_{L^p(\Delta,\nu_{H,\Delta})}^{1-\epsilon} \|g\|_\infty^\epsilon C_{m,p}^\epsilon \langle x - y\rangle^{-m\epsilon}$$

for the Borel functions $g = f$ or $g = \chi$. By assumption on the Hölder continuity $\|\chi\|_{L^p(\Delta,\nu_{H,\Delta})} \leq c\,|\Delta'|^{\frac{\gamma}{p}}$. Choosing m large enough and combining the polynomial decays yields the stated result. □

Corollary 4.3.15 *Under the conditions of Proposition 4.3.14 H satisfies a fractional moments bound for any $0 < s < \gamma$.*

Proof. We prove the estimate for $z = \imath\mu \in \imath\mathbb{R} \setminus \{0\}$ since it is trivial to shift the real part and further assume w.l.o.g. that $\Delta = [-\frac{1}{2}, \frac{1}{2}]$. The strategy for the estimate is to decompose

$$(H + \imath\mu)^{-1} = \sum_{j=1}^{\infty} (H + \imath\mu)^{-1} \chi_j(H)$$

into the dyadic pieces $\chi_j(\lambda) = \chi(|\lambda| \in [2^{-l-1}, 2^{-l}))$. Then

$$\left\|P_x (H + \imath\mu)^{-1} \varphi(H) P_y\right\|_s^s \leq \sum_{j=1}^{\infty} \left\|P_x (H + \imath\mu)^{-1} \chi_j(H) \varphi(H) P_y\right\|_s^s$$

and since $\left\|(H+\imath\mu)^{-1}\chi_j(H)\varphi(H)\right\|_\infty \leq c_1 2^j$ uniformly in μ the Proposition gives

$$\left\|(H+\imath\mu)^{-1}\chi_j(H)\varphi(H)\right\|_s^s \leq c_2 2^{sj}\, 2^{-j\gamma(1-\epsilon)}\langle x-y\rangle^{-ks}.$$

The sum over j is absolutely convergent provided $\gamma(1-\epsilon) > s$ (which can always be satisfied by choosing ϵ small enough due to $\gamma > s$) and then

$$\left\|P_x(H+\imath\mu)^{-1}\varphi(H)P_y\right\|_s^s \leq c_3\langle x-y\rangle^{-ks}$$

uniformly in μ as we wanted to prove. The same argument applies to the resolvent of the bounded transform. □

4.3.2 Pseudogaps

We say that a Hamiltonian H has a pseudogap if there exists a point E_0 in the spectrum at which the density of states vanishes like some fractional power of the energy. In particular, if the DOS-measure is absolutely continuous then its density should behave as

$$\left|\frac{\mathrm{d}\nu_{\Delta,H}}{\mathrm{d}E}(E)\right| \;\leq\; C\,|E-E_0|^{\gamma-1}$$

for some $\gamma > 1$. More generally we do not require absolute continuity:

Definition 4.3.16 *Let H be a self-adjoint $\mathcal{A}$-multiplier with Δ a $\mathcal{T}$-finite spectral region. If there exists a point $E_0 \in \Delta$ where the density of states of is γ-Hölder continuous with $\gamma > 1$, i.e.*

$$\nu_{\Delta,H}((E_0-\epsilon, E_0+\epsilon)) \leq C\epsilon^\gamma$$

for small enough ϵ, then we say that H has a pseudogap at E_0 of order γ.

It has been shown in [111] that in a von Neumann algebra with finite trace a pseudogap of positive order implies that the (unbounded) inverse $(H-E_0)^{-1}$ lies in certain L^p-spaces. In the general semi-finite case this can at most be the case for a local inverse:

Proposition 4.3.17 (cf. [111, Proposition 5.3.8]) *Assume that the DOS of the self-adjoint $\mathcal{A}$-multiplier H is well-defined in a spectral interval Δ and γ-Hölder continuous at E_0 with $\gamma > 0$. For $\kappa \in (0,1)$ introduce the imaginary double-cone*

$$D_{M,\kappa} \;=\; \left\{Re^{\imath\theta} \in \mathbb{C} \;:\; \left|R\cos(\theta)\right| \leq \tfrac{M}{2} \;\text{ and }\; \left|\cos(\theta)\right| \leq \kappa\right\}.$$

(i) For $e_\Delta = \chi(H \in \Delta)$ there is a local inverse $\frac{e_\Delta}{H-E_0+z} \in L^p(\mathcal{A})$ for all $p \in (0,\gamma)$ for which

$$\frac{e_\Delta}{H-E_0+z}e_\Delta(H-E_0+z) = e_\Delta = e_\Delta(H-E_0+z)\frac{e_\Delta}{H-E_0+z}$$

and its L^p-norm is bounded uniformly for all $z \in D_{M,\kappa}$.

(ii) For all $p \in (0,\gamma)$ and $r \in (0,\gamma-p)$, one has

$$\left\|\frac{e_\Delta}{H-E_0} \;-\; \frac{e_\Delta}{H-E_0+z}\right\|_p \;\leq\; K_\kappa\left|\Im m(z)\right|^s \tag{4.3.4}$$

with $s = \min\{\frac{r}{p},1\}$ and a constant K_κ uniformly for all $z \in D_{M,\kappa}$.

Proof. The γ-Hölder continuity implies

$$\mathcal{T}(\chi(H=E_0)) \leq \lim_{\epsilon\downarrow 0}\nu_h([E_0-\epsilon,E_0+\epsilon]) = 0$$

and since $\mathcal{T}$ is a faithful trace E_0 is therefore not an eigenvalue of H. Since one can easily rescale H as needed we assume without loss of generality that $E_0 = 0$ and $\Delta = [-1,1]$.

(i) As 0 is not an eigenvalue the local inverse is well-defined using unbounded functional calculus and we only need to bound L^p.norms of $e_\Delta H^{-1}$. One notes relation

$$\chi_{[a,b]}(|H|^{-1}) \;=\; \chi_{[\frac{1}{b},\frac{1}{a}]}(|H|) \;=\; \chi_{[-\frac{1}{a},-\frac{1}{b}]}(H) \;+\; \chi_{[\frac{1}{b},\frac{1}{a}]}(H)\,.$$

To make use of the density of states we decompose the interval $\Delta = [-1, 1]$ into dyadic components to estimate

$$\begin{aligned}\mathcal{T}\left(e_\Delta\,|H|^{-p}\right) \;&\leq\; \sum_{k=0}^{\infty} 2^{kp}\nu_{\Delta,H}([-2^{-k},-2^{-k-1})\cup(2^{-k-1},2^{-k}]))\\ &\leq\; C\sum_{k=0}^{L} 2^{k(p-\gamma)} \;<\; \infty\,.\end{aligned}$$

Hence $e_\Delta\,|H|^{-1} \in L^p(\mathcal{A})$ and to get a uniform bound for $z \in D_{M,\kappa}$ one shifts with the resolvent identity

$$\begin{aligned}\mathcal{T}\left(e_\Delta\left|\frac{1}{H+z}\right|^p\right) \;&=\; \int_{-1}^{1}\frac{1}{|\lambda+z|^p}\,\nu_{\Delta,H}(\mathrm{d}\lambda)\\ &\leq\; \int_{-1}^{1}\left(1+\frac{|z|}{|\Im m(z)|}\right)^p\frac{1}{|\lambda|^p}\,\nu_{\Delta,H}(\mathrm{d}\lambda)\\ &\leq\; \int_{-1}^{1}\left(1+(1-\kappa^2)^{-\frac{1}{2}}\right)^p\frac{1}{|\lambda|^p}\,\nu_{\Delta,H}(\mathrm{d}\lambda)\\ &\leq\; \left(1+(1-\kappa^2)^{-\frac{1}{2}}\right)^p\,\mathcal{T}\left(e_\Delta\left|\frac{1}{H}\right|^p\right).\end{aligned}$$

The final line used the two elementary estimates

$$\frac{1}{\lambda+z} = \left(1-\frac{z}{\lambda+z}\right)\frac{1}{\lambda}, \qquad \frac{|z|}{|\Im m(z)|} < (1-\kappa^2)^{-\frac{1}{2}}.$$

(ii) For $r \geq p$ the estimate is a simple consequence of the Hölder inequality

$$\left\|\frac{e_\Delta}{H}-\frac{e_\Delta}{H+z}\right\|_p \;=\; |z|\,\left\|\frac{e_\Delta}{H+z}\,\frac{e_\Delta}{H}\right\|_p \;\leq\; |z|\,\left\|\frac{e_\Delta}{H+z}\right\|_{2p}\,\left\|\frac{e_\Delta}{H}\right\|_{2p}\,,$$

since $\gamma > 2p$. For $r < p$ one applies to $a \in L^{p+r}(\mathcal{A}) \cap L^\infty(\mathcal{A})$ and $b \in L^p(\mathcal{A}) \cap L^{p+r}(\mathcal{A})$ first the Hölder inequality and then log-convexity (1.3.2) of the p-norms to bound

$$\|ab\|_p \;\leq\; \|a\|_{p(1+\frac{p}{r})}\,\|b\|_{p+r} \;\leq\; \|a\|_{p+r}^{\frac{r}{p}}\,\|a\|_\infty^{1-\frac{r}{p}}\,\|b\|_{p+r}\,. \qquad (4.3.5)$$

In the situation here this gives

$$\left\|\frac{e_\Delta}{H} - \frac{e_\Delta}{H+z}\right\|_p = |z| \left\|\frac{e_\Delta}{H+z}\,\frac{e_\Delta}{H}\right\|_p \le |z| \left\|\frac{e_\Delta}{H+z}\right\|_\infty^{1-\frac{r}{p}} \left\|\frac{e_\Delta}{H+z}\right\|_{p+r}^{\frac{r}{p}} \left\|\frac{e_\Delta}{H}\right\|_{p+r}$$
$$\le \frac{|z|}{|\Im m(z)|^{1-\frac{r}{p}}} \left\|\frac{e_\Delta}{H+z}\right\|_{p+r}^{\frac{r}{p}} \left\|\frac{e_\Delta}{H}\right\|_{p+r}.$$

As $\frac{|z|}{|\Im m(z)|} < (1-\kappa^2)^{-\frac{1}{2}}$ for $z \in D_{M,\kappa}$ and $p + r < \gamma$ item (i) completes the proof.
□

With the bound of Proposition 4.3.17 we can control holomorphic approximations to the Fermi projection if the pseudogap occurs exactly at the Fermi level:

Proposition 4.3.18 *If H has a pseudogap at $E_F \in \Delta$ of order $\gamma = mp + \delta$ with $p \ge 1$, $m \in \mathbb{N}_+$ and $\delta > 0$ and is a p-smooth perturbation of the reference Hamiltonian H_0 then*

$$e_F - e_0 \in W_p^m(\mathcal{A}).$$

Proof.

Since E_F cannot be an eigenvalue we may use the representation of e_F as a Riesz projection for $F(H)$ from Lemma A.6. Without loss of generality $E_F = 0$ and with a smooth function φ supported in Δ we write

$$\begin{aligned} e_F - e_0 &= \operatorname*{s-lim}_{\epsilon\to 0} \int_{\mathcal{C}_\epsilon^-} (F(H)+z)^{-1}(F(H)-F(H_0))(F(H_0)+z)^{-1}\mathrm{d}z \\ &= \operatorname*{s-lim}_{\epsilon\to 0} \int_{\mathcal{C}_\epsilon^-} (F(H)+z)^{-1}(1-\varphi(H))(F(H)-F(H_0))(F(H_0)+z)^{-1}\mathrm{d}z \\ &+ \operatorname*{s-lim}_{\epsilon\to 0} \int_{\mathcal{C}_\epsilon^-} (F(H)+z)^{-1}\varphi(H)(F(H)-F(H_0))(F(H_0)+z)^{-1}\mathrm{d}z \end{aligned}$$

The term containing $(1-\varphi(H))$ is uniformly smooth due to the cutoff and the invertibility of H_0, hence it is easy to see that the integral converges in the topology of $W_p^\infty(\mathcal{A})$ with seminorms bounded uniformly in ϵ. The difficult part is therefore the remainder

$$\int_{\mathcal{C}_\epsilon^-} (F(H)+z)^{-1}\varphi(H)(F(H)-F(H_0))(F(H_0)+z)^{-1}\mathrm{d}z.$$

From the resolvent bound of Proposition 4.3.17 it is clear that

$$\begin{aligned}&\left\|(F(H)+z)^{-1}\varphi(H)(F(H)-F(H_0))(F(H_0)+z)^{-1}\right\|_p\\&\leq \left\|(F(H))^{-1}\varphi(H)\right\|_p \, \|F(H)-F(H_0)\| \, \left\|(F(H_0))^{-1}\right\| < \infty\end{aligned}$$

and that the integral is therefore bounded for $\epsilon \to 0$.

To complete the proof it is sufficient to bound for each finite $\epsilon > 0$ and $|j| \leq n$ the p-norm of the integral

$$\int_{\mathcal{C}_\epsilon^-} \nabla^j \left((F(H)+z)^{-1}(F(H)-F(H_0))(F(H_0)+z)^{-1}\right) \mathrm{d}z.$$

Via the Leibniz rule we can expand the integrand as a sum of terms of the form

$$\left(\prod_{i_1=1}^{|j_1|}(F(H)+z)^{-1}c_{i_1}\right)(F(H)+z)^{-1}B(F(H_0)+z)^{-1}\left(\prod_{i_2=1}^{|j_2|} d_{i_2}(F(H_0)+z)^{-1}\right)$$

where B is a derivative of $F(H)-F(H_0)$ and the c_i,d_i are derivatives of $F(H)$ and $F(H_0)$ respectively. The resolvents of $F(H_0)$ are uniformly bounded on the contours $\mathcal{C}_\epsilon^-$ and therefore unproblematic. We decompose each occurence of the resolvent as

$$(F(H)+z)^{-1} = (F(H)+z)^{-1}\varphi(H) + (F(H)+z)^{-1}(1-\varphi(H))$$

and multiply out. The factors $(1-\varphi(H))$ have uniformly bounded operator norm and it is sufficient to explain how to deal with the extreme cases. The one extreme is where one only has factors $(1-\varphi(H))$ in which case the product can be bounded by $\|B\|_p$ times a constant. The the other extreme is where one has $|j_1|$ factors of $\varphi(H)$ and thus one has to deal with

$$\left(\prod_{i_1=1}^{|j_1|}\frac{1}{F(H)-z}\varphi(H)c_{i_1}\right)(F(H)+z)^{-1}\varphi(H)B\, g(z)$$

where $|j_1| \leq m$ and g is an operator-norm bounded function of z. Choosing any $r = \frac{\delta}{2}$ log-convexity (1.3.2) and the standard resolvent estimate allow us to bound

$$\begin{aligned}\left\| \prod_{i_1=1}^{|j_1|+1} \frac{1}{F(H)-z}\varphi(H)c_{i_1} \right\|_p &\leq c \left\| \frac{1}{F(H)-z}\varphi(H) \right\|_{(|j_1|+1)p}^{|j_1|+1} \\ &\leq c \left\| \frac{1}{F(H)-z}\varphi(H) \right\|_{\infty}^{\frac{p-r}{p}} \left\| \frac{1}{F(H)-z}\varphi(H) \right\|_{|j_1|p+r}^{\frac{|j_1|p+r}{p}} \\ &\leq c \left|\Im m(z)\right|^{-\frac{p-r}{p}} \left\| \frac{1}{F(H)-z}\varphi(H) \right\|_{|j_1|p+r}^{\frac{|j_1|p+r}{p}} \end{aligned} \qquad (4.3.6)$$

and that final norm is uniformly bounded on all curves C_ϵ due to the DOS-estimate Proposition 4.3.17(ii).

□

Examples of Hamiltonians with pseudogap will be given later. Let us remark that if the resolvents of H,H_0 lie in $L^p(\mathcal{A})$ then there is a close analogue to Proposition 4.3.10 for paths of Hamiltonians with a common pseudogap, i.e. automatic continuity of the Fermi projections in some Sobolev norms. For a proof, we note that for χ_ϵ the function from Lemma A.6 one can similarly as above estimate the difference of $\chi_\epsilon(F(H)) - e_F$ in Sobolev norm using the pseudogap and then continuity follows similarly as in the proof of Proposition 4.3.10 using an $\frac{\epsilon}{3}$-argument. In the case of a finite trace one does not even have to assume that the path itself is smooth, for automatic continuity a uniform mobility or pseudogap is sufficient (see [100, Proposition 6] or [111, Proposition 5.3.13]).

4.3.3 Index theorems

With those perturbative results in place there are well-defined relative pairings between H and H_0 with Chern cocycles:

Theorem 4.3.19 *Let H, H_0 be θ-smooth self-adjoint $\mathcal{A}$-multipliers. Assume that H_0 has a spectral gap Δ and that H is a p-smooth perturbation of the reference Hamiltonian H_0. Let $\alpha : \mathbb{R}^n \times \mathcal{A} \to \mathcal{A}$ be a restriction of θ to an n-parameter subgroup. If either* *(i) H also has a spectral gap in* Δ*,*

(ii) H has a mobility gap in Δ*,*

(iii) H has a pseudogap of order $\gamma > p$ *in* $E_F \in \Delta$*,*

for some $p \in (n, n+1]$ *for* $n \in \mathbb{N}$ *then we have well-defined relative pairings*

- *for n odd assume the CH is satisfied and set*

$$\langle Ch_{\mathcal{T},\alpha}, [u_F u_0^*]_1 \rangle \in \mathbb{R}$$

with the Fermi unitaries (u_F, u_0) *of* (H, H_0) *and which can be in the notation of Theorem 2.3.5 written as an index over* $L^\infty(\mathcal{A}) \rtimes_\alpha \mathbb{R}^n$ *as*

$$\langle Ch_{\mathcal{T},\alpha}, [u_F u_0^*]_1 \rangle = \hat{\mathcal{T}}_\alpha\text{–}\mathrm{Ind}(\mathbf{P}_{x_0} u_F u_0^* \mathbf{P}_{x_0} + 1 - \mathbf{P}_{x_0}).$$

- *for n even set*

$$\langle Ch_{\mathcal{T},\alpha}, [e_F]_0^M - [e_0]_0^M \rangle = \langle Ch_{\mathcal{T}^s,\alpha\times\lambda}, [\hat{u}_F \hat{u}_0^*]_1 \rangle \in \mathbb{R}$$

with the suspensions of the Fermi projections $\hat{u}_* = f(D)e_* + 1 - e_*$ *of H respectively* H_0 *and notations as in Proposition 2.4.2. If either (i) or (ii) holds, or additionally* (iii) *with* $\gamma > n+1$*, then it can also be written as an index*

$$\hat{\mathcal{T}}^s_{\alpha\times\lambda}\text{–}\mathrm{Ind}(\tilde{\mathbf{P}}_{x_0} \hat{u}_F \hat{u}_0^* \tilde{\mathbf{P}}_{x_0} + 1 - \tilde{\mathbf{P}}_{x_0}) = \langle Ch_{\mathcal{T}^s,\alpha\times\lambda}, [\hat{u}_F \hat{u}_0^*]_1 \rangle$$

as in Corollary 2.4.3.

In the case (i) those are pairings with classes in $K_1(\mathcal{A})$ *respectively* $K_1(S\mathcal{A}) \simeq K_0(\mathcal{A})$*, otherwise they are pairings with the K-theory of any* C^**-subalgebra of* $L^\infty(\mathcal{A})$ *respectively* $L^\infty(\mathbb{R}) \otimes L^\infty(\mathcal{A})$ *which contains the respective elements.*

Proof. In the odd case the perturbative results Proposition 4.3.9 respectively Proposition 4.3.18 yield $u_F u_0^* \in W_p^1(\mathcal{A})$ with $p \in (n, n+1]$, hence Theorem 2.3.5 applies. Similarly, in the even case the conditions for Proposition 2.4.2 are satisfied and under the stated additional conditions also those for Corollary 2.4.3. □

As one can see, in the pseudogapped case a bit more regularity is required for the index theorem than to just define the pairing with the culprit being the suspension which increases the necessary L^p-regularity (in the spectrally gapped or mobility gapped case this additional regularity is always present automatically). The numerical evaluation of the pairings proceeds as described in Section 2.4, namely using suspensions in the even case. In concrete situations it is sometimes possible to also evaluate those relative pairings as formal differences of Chern numbers

$$\sum_{\rho\in S_n}(-1)^{\rho}\,(\mathcal{T}\otimes\mathrm{Tr})\big(e_F\nabla_{\rho(1)}e_F\cdots\nabla_{\rho(n)}e_F-e_0\nabla_{\rho(1)}e_0\cdots\nabla_{\rho(n)}e_0\big)$$

with the Fermi projection e_F, e_0 in the even and

$$\sum_{\rho\in S_n}(-1)^{\rho}\,(\mathcal{T}\otimes\mathrm{Tr})\big(u_F^*\nabla_{\rho(1)}u_F\cdots\nabla_{\rho(n)}u_F-u_0^*\nabla_{\rho(1)}u_0\cdots\nabla_{\rho(n)}u_0\big)$$

with the Fermi unitaries u_F,u_0 in the odd case. In the spectrally gapped case there are always representatives for which the formulas are true by Proposition 3.4.4, but outside of that regime e.g. in a mobility gap it is a quite subtle problem to establish that these formulas are the same as the indices of Theorem 4.3.19 whenever they make sense (whether in terms of L^p-regularity or under even weaker regularizations).

If the Fermi projection/unitary by itself is already a matrix over the unitization of a Sobolev space of sufficient regularity then one can eliminate the reference Hamiltonian and write down an index pairing in terms of H alone.

Definition 4.3.20 *We say that a θ-smooth self-adjoint $\mathcal{A}$-multiplier H is strongly p-smooth if $F(H)$ is a matrix over $W_p^\infty(\mathcal{A})^\sim$.*

A strongly p-smooth reference Hamiltonian also has a Fermi projection which is a matrix over $M_N(\mathcal{A}^\sim)$:

Proposition 4.3.21 *Let H be a strongly p-smooth Hamiltonian with spectral gap Δ then the associated Fermi projection $\chi(H<\Delta)$ is also a matrix over $W_p^\infty(\mathcal{A})^\sim$.*

Proof. The resolvent $(F(H)+z)^{-1}$ for a strongly p-smooth Hamiltonian lies in $M_N(L^p(\mathcal{A})^\sim)$ since $L^p(\mathcal{A})\cap\mathcal{A}$ is spectrally invariant in $\mathcal{A}$. Subtracting the scalar part $s: M_N(\mathcal{A}^\sim)\to M_N(\mathbb{C})$ does not affect commutators with X_i, e.g.

$$\nabla\left((F(H)+z)^{-1}-s\left((F(H)+z)^{-1}\right)\right)=(F(H)+z)^{-1}(\nabla F(H))(F(H)+z)^{-1}$$

which implies that $(F(H)+z)^{-1}-s\left((F(H)+z)^{-1}\right)$ lies in $W_p^\infty(\mathcal{A})$ with operator norm bounded in terms of a power of $\mathrm{dist}(z,\sigma(H))$. Due to the spectral gap one can write $\chi(H < \Delta)$ as a Riesz projection in terms of resolvents $(F(H) + z)^{-1}$ which makes it easy to complete the proof. □

The results derived above immediately give criteria for when the perturbations of a strongly p-smooth Hamiltonian are again strongly p-smooth. They need to be seeded with the strong p-smoothness of some reference Hamiltonian H_0 which in many cases can be asserted via an explicit diagonalization. In the easiest case the reference Hamiltonian is bounded from below:

Proposition 4.3.22 *Let H be a θ-smooth self-adjoint $\mathcal{A}$-multiplier which admits a lower bound $H > m$ for some $m \in \mathbb{R}$. Assume that $(H + \imath)^{-1} \in L^p(\mathcal{A})$ for all $p \in (p_0, \infty]$, then H is strongly p-smooth for all $p \in (p_0, \infty]$.*

Proof. Due to the lower bound we can write $F(H) = 1 - g(H)$ with a function g that decays as $O(\lambda^{-2})$ at $\pm\infty$. We therefore have the resolvent representation

$$F(H) - 1 = \frac{1}{2\pi} \int_{\mathbb{C}} (\partial_{\overline{z}} \tilde{g}_K)(z) \frac{1}{H - z} \mathrm{d}z \wedge \mathrm{d}\overline{z}$$

and applying the log-convexity (1.3.2)

$$\begin{aligned} \left\| \frac{1}{H-z} \right\|_p &\leq \left\| \frac{1}{H-z} \right\|_{p(1-\epsilon)}^{1-\epsilon} \left\| \frac{1}{H-z} \right\|_{\infty}^{\epsilon} \\ &\leq |\Im m\, z|^{-\epsilon} \left\| \frac{1}{H+\imath} \right\|_{p(1-\epsilon)}^{1-\epsilon} \left(1 + \frac{\langle \Re e\, z \rangle}{|\Im m z|} \right)^{(1-\epsilon)} \end{aligned}$$

the integral is seen to converge in L^p for each $p \in (p_0, \infty]$ by choosing ϵ so small that $p(1-\epsilon) > p_0$ and applying Lemma A.4 with the knowledge that $g \in \mathcal{S}^{\beta}(\mathbb{R})$ for all $\beta < 2$.

Similarly one checks that the derivatives exist by differentiating under the integral. □

To construct a strongly p-smooth Hamiltonian which is not bounded from below one can take, for example, some H which is bounded from below and set $\tilde{H} = \sigma \otimes H$ for a self-adjoint unitary σ which is also strongly p-smooth since $F(\tilde{H}) = \sigma \otimes F(H)$. From then one can argue via perturbations and add p-smooth potentials or certain unbounded perturbations.

Once one adds to a strongly p-smooth reference Hamiltonian (random) potentials which lead to spectral gaps, mobility gaps or pseudogaps one can assert via the above perturbative results that one still has a Fermi projection which is at least scalar matrix over a Sobolev space. For such a perturbation of a strongly p-smooth Hamiltonian one can define its Chern numbers without a reference Hamiltonian simply via the pairing with the Chern cocycles. They are then protected by the semi-finite index theorems of Section 2.3:

Theorem 4.3.23 *Let H, H_0 be θ-smooth self-adjoint $\mathcal{A}$-multipliers. Assume that H_0 has a spectral gap Δ, is strongly p-smooth and that H is a p-smooth perturbation of the reference Hamiltonian H_0. Let $\alpha : \mathbb{R}^n \times \mathcal{A} \to \mathcal{A}$ be a restriction of θ to an n-parameter subgroup. If either*

(i) H also has a spectral gap in Δ,

(ii) H has a mobility gap in Δ,

(iii) H has a pseudogap of order $\gamma > p$ in $E_F \in \Delta$,

for some $p \in (n, n+1]$ for $n \in \mathcal{N}$ and the CH is satisfied for odd n then we have the index pairings

$$\hat{\mathcal{T}}_\alpha\text{–Ind}(\mathbf{P}u_F\mathbf{P} + 1 - \mathbf{P}_\alpha) = \langle Ch_{\mathcal{T},\alpha}, [u_F]_1 \rangle \in \mathbb{R}$$

in the odd case with the Fermi unitary u_F of H respectively

$$\hat{\mathcal{T}}_\alpha\text{-Ind}(e_F \mathbf{G}_{x_0} e_F + 1 - e_F) = \langle Ch_{\mathcal{T},\alpha}, [e_F]_0 \rangle$$

with the Fermi projection $e_F = \chi(H < E_F)$. In the case (i) those are pairings with classes in $K_{n \mod 2}(\mathcal{A})$.

Proof. Due to Proposition 4.3.21 and Proposition 4.3.9 respectively Proposition 4.3.18 one has $e_F \in M_N(W_p^1(\mathcal{A}))$, which implies $u_F \in M_{N/2}(W_p^1(\mathcal{A}))$ in the odd case. Hence one can apply Theorem 2.3.5. □

The reference Hamiltonian does not appear in the index pairings anymore, it is only used to assert regularity of the Fermi projection. The relative indices of Theorem 4.3.19 are simply the differences between the respective absolute indices for H and H_0 (to see this one notes that both are pairings with the K-theory of a Banach algebra $K_i(W_p^1(\mathcal{A}) \cap L^\infty(\mathcal{A}))$).

Let us discuss stability properties of those indices both in the relative and absolute case. Being semi-finite Fredholm indices they are invariant under norm-continuous homotopies of the representatives as long as the Fredholm property is preserved. In the presence of a spectral gap correct notion of homotopy of a pair (H, H_0) that preserves the K-theory classes is that

- (H, H_0) change continuously in the Riesz-topology (i.e. $(F(H), F(H_0))$ changes continuously),
- the constraint $F(H) - F(H_0) \in M_N(\mathcal{A})$ is satisfied,
- the common spectral gap Δ of H,H_0 does not close.

Those conditions imply that the K-theory class $[e_F]_0^M - [e_0]_0^M$ respectively $[u_F]_1^M - [u_0]_1^M$ does not change (for the latter one also requires that the chiral symmetry remains intact).

In the absence of a spectral gap for H, however, bounded perturbations of the Hamiltonian will not result in norm-continuous perturbations of the Fermi projection; one only expects continuity in the weaker operator topologies ones. As long as one has a uniform mobility gap along a deformation the relative (and absolute) Chern numbers must nevertheless change continuously due to Proposition 4.3.10 and the continuity of the Chern cocycles. There is some evidence that in physically relevant situations the Chern numbers will often be invariant under homotopy also in the mobility gapped case even if there is no immediate K-theoretic reason for that. One can make the argument rigorous for some special algebras including $C(\mathbb{T}^d_{0,\Omega})$ under a slightly stronger mobility gap assumption, as will be sketched in Section 4.3.4.

A similar continuity result is true for pseudogapped Hamiltonians. One must, however, take the term topological invariant with a grain of salt, since the Chern numbers will at most be invariant under a restrictive class of homotopies and in the case of a pseudogap vary freely under perturbations. Only the fact that a Hamiltonian does have non-trivial Chern numbers at all is perturbatively stable. This is sufficient for some purposes since the precise values do not matter for the expected phenomenology, e.g. any non-vanishing value predicts the occurrence of boundary states.

On another note, the index theorems show that the Chern numbers still make sense and are stably for Hamiltonians affiliated to the algebra $L^\infty(\mathcal{A} \rtimes_\theta \mathbb{R}^d)$. This can be used e.g. to extend bulk-boundary correspondence to certain non-smooth Hamiltonians that still have a spectral gap (see [111, Proposition 5.4.5] for an example).

Let us finally comment on the values of the indices. A priori they can take values in all of $\mathbb{R}$, however, if $\mathcal{A}$ is separable and thus has countably generated K-groups then the possible values must also be countable in the spectrally gapped case though they may be dense in $\mathbb{R}$. Depending on the algebra $\mathcal{A}$ they can take largely arbitrary values. For the algebras $C(\mathbb{T}^d_{\mathbf{B},\Omega})$ and $C(\mathbb{R}^d_{\mathbf{B},\Omega})$ the strong Chern numbers (i.e. those which involve all d generators $\alpha = \theta$) are always integers. The reason is that the auxiliary von Neumann algebra $L^\infty(\mathcal{A} \rtimes_\theta G)$ of Chapter 2 is then a doubly crossed products and can be written as a type-I_∞ algebra due to Takesaki-duality Theorem 1.1.6. Thus the semifinite Fredholm modules of Section 2.3 decompose as direct integrals over some measure space $(\Omega, \mathbb{P})$. The semifinite index of some $\int_\Omega^\oplus T_\omega \mathrm{d}\mathbb{P}(\omega)$ is then just the average over the (usual Hilbert-space Fredholm) indices of its fibers $\int_\Omega \mathrm{Ind}(T_\omega)\mathrm{d}\mathbb{P}(\omega)$. Since we assume ergodicity for Ω the index of T_ω almost surely does not depend on ω, thus the average can only take values $\mathbb{Z}$. Examples where this process has been worked out (also for half-space algebras) can be found in e.g. [74, 103, 111]. Combined with the continuity of the Chern numbers asserted above, this implies in particular that integer-valued Chern numbers must also be exactly constant in the mobility gapped regime.

4.3.4 Examples: Tight-binding models

We begin with the example of the non-commutative torus $\mathcal{A} = M_N(C(\mathbb{T}^d_{\mathbf{B},\Omega}))$. Since it is unital one can as reference Hamiltonian always fix $H_0 = \mathbb{1}$ or $H_0 = \sigma_1$ with σ_1 some matrix that anti-commutes with J (to stay in the chiral symmetry class).

Therefore, any spectrally gapped Hamiltonian $H \in M_N(\mathcal{A})$ defines canonically a K-theory class in $K_i(\mathcal{A})$ which is (for contractible disorder space) completely determined by the set of weak Chern numbers

$$\langle \mathrm{Ch}_{\mathcal{T},v_1\times\ldots\times v_n}, [e_F]_0\rangle, \quad \langle \mathrm{Ch}_{\mathcal{T},v_1\times\ldots\times v_n}, [u_F]_1\rangle,$$

where $v_1 \times \ldots \times v_n$ denotes the $\mathbb{R}^n$-action obtained by restricting θ to span of the orthonormal vectors $v_1, \ldots, v_n$. The countably many possible values of those pairings can then also be computed in terms of the magnetic field $\mathbf{B}$ [103]. For large enough disorder (which preserves the chiral symmetry if there is one) there will often not be an actual spectral gap in the spectrum but only a mobility gap in the sense of Definition 4.3.8. In that case all weak Chern numbers are still well-defined via the Chern cocycles for Sobolev spaces. Let us now sketch an argument that at least for the non-commutative torus the Chern numbers will often stay constant under perturbations that preserve a fixed mobility gap, even though the K-theory class of the Fermi projection changes.

The idea is based on the fact that if the disorder space is a product space $\Omega = \Omega_0^{\mathbb{Z}^d}$ and the magnetic field $\mathbf{B}$ is rational then one has well-defined finite volume approximations [99], i.e. $C(\mathbb{T}^d_{\mathbf{B},\Omega})$ is a projective limit

$$C(\mathbb{T}^d_{\mathbf{B},\Omega}) = \varprojlim_{\Lambda\to\infty} \pi_\Lambda(C(\mathbb{T}^d_{\mathbf{B},\Omega}))$$

where π_Λ describes finite-volume representations of $C(\mathbb{T}^d_{\mathbf{B},\Omega})$ on a finite lattice Λ with periodic boundary conditions. While the range of π_Λ is not finite-dimensional it consists of finite-size matrices over $C(\Omega_\Lambda)$ for the product space $\Omega_\Lambda = (\Omega_0)^\Lambda$. In particular, the Chern numbers are well-approximated by rapidly converging finite-volume approximations, hence they are also computable in practice. The lattices Λ do not actually have to be finite, one can also use semi-infinite strips on slabs of shape $\Lambda_{n,L} := \mathbb{Z}^n \times [-L, L]^{d-n}$ with periodic boundary conditions in the finite directions. Let $(h_t)_{t\in[0,1]}$ be a smooth family of smooth Hamiltonians with a common mobility gap and assume that furthermore one has a fractional moments bound that holds uniformly for periodic approximations, for example

$$\sup_\Lambda \sup_{t\in[0,1]} \sup_{x,y\in\Lambda} e^{\mu|x-y|_\Lambda} \int_{\Omega_\Lambda} \left| \langle x| \frac{1}{\pi_{\Lambda,\omega}(h_t) + z} |y\rangle \right|^s \mathrm{d}\mathbb{P}_\Lambda(\omega) < \infty$$

for some $\mu > 0$, $0 < s < 1$, z as in Definition 4.3.11 and where $|x-y|_\Lambda$ is the ℓ^1-distance on Λ which takes periodicity into account. Let us now consider the stability of the n-parameter Chern number Ch_n for directions $e_1, ..., e_n$. Let us claim that the finite-volume error in

$$\langle \mathrm{Ch}_n, [\chi(h_t < E_F)]_0\rangle = \lim_{L\to\infty} \langle \mathrm{Ch}_{L,n}, [\chi(\pi_{\Lambda_{n,L}}(h_t) < E_F)]_0\rangle \tag{4.3.7}$$

can then be controlled with bounds independent of t. However, for the smooth family $\pi_{\Lambda_{n,L}}(h_t)$ of quasi-n-dimensional mobility gapped insulators the corresponding n-dimensional Chern number is a strong Chern number and therefore can only take quantized values in $c_L\mathbb{Z}$ due to ergodicity (though on a scale $c_L = O(L^{-(d-n)})$). Therefore the right-hand side of (4.3.7) is a limit of t-independent constants. In combination with the uniform error bound this implies that the infinite-volume weak Chern numbers must all be constant on such a smooth path. A detailed argument will be presented elsewhere since it is lengthy and only applies in a rather special setup. Let us also point out that in contrast the K-theory class of the Fermi-projection in any of the Sobolev algebras $(W^1_p \cap L^\infty)(\mathbb{T}^d_{\mathbf{B},\Omega})$ will practically never be unchanged under such a deformation since the density of states $\mathcal{T}(\chi(h_t < E_F))$ is not quantized in a mobility gap. This is no contradiction to the claim above, the argument above also fails for the 0-dimensional Chern number since it is not quantized as an integer in finite volume approximations (due to the averaging over the disorder space which takes place).

We can also give an example of a Hamiltonian with a pseudogap and nonvanishing weak Chern numbers which arises naturally as a tight-binding model for graphene in the semimetal phase [103]

$$H = \begin{pmatrix} 0 & 1 + u_1 + u_1 u_2^* \\ 1 + u_1^* + u_1^* u_2 & 0 \end{pmatrix}.$$

For that model one has a linear pseudogap and the non-trivial one-dimensional weak Chern number $\mathrm{Ch}_1(u_F) = \frac{1}{3}$. This number is highly sensitive to perturbations, if one changes the hopping coefficients of the model continuously without opening a gap it will change continuously. A similar example for a three-dimensional Weyl semimetal with non-trivial two-dimensional weak Chern numbers will be given further below in Section 6.3. There is apparently no realistic model of a disordered semimetal which is rigorously known to have a pseudogap and numerical computations are ambiguous. Regular diagonal disorder is known to not allow pseudogaps [65] and there are heuristic arguments that any pseudogap is filled up by disorder due to so-called rare-region effects [90]. In any case, if one could add pseudogap-preserving disorder to the model above then it would have to have a non-vanishing weak Chern number at small disorder. Let us also note that it may be easier to find such disorder not in the class of random ergodic perturbations, but rather in the quasi-periodic ones [84].

4.3.5 Examples: Tight-binding models (multipliers)

Another application of the reference Hamiltonian formalism are substrates. In practice, a two-dimensional topological insulator will not exist in a free form, but is e.g. supported on a three-dimensional substrate and one cannot avoid at least a weak coupling. For another example, in the quantum Hall effect the system under consideration is usually not an actual two-dimensional electron gas, but rather a thin interface layer between two semiconductors. While one may of course argue that an effective two-dimensional description is perfectly sufficient, one may wonder if the coupling to the bulk material can affect the topological classification. With the reference Hamiltonian approach one can see that in fact this is not the case as long as the combined system consisting of surface layer and substrate has a spectral gap.

We set $\mathcal{A} = C(\mathbb{T}^d_{\mathbf{B},\Omega}) \otimes \mathbb{K}(\ell^2(\mathbb{N}))$ and note that this algebra can be represented as acting on $\ell^2(\mathbb{Z}^d \times \mathbb{N})$ with a dense subset given by those observables which act only on finite slabs $\mathbb{Z}^d \times [0, L]$. One may think of $\mathcal{A}$ as an algebra of observables which sits on the surface of a $(d+1)$-dimensional system.

Let us choose a reference Hamiltonian $H_0 \in M(\mathcal{A})$ which lives on the halfspace $\mathbb{Z}^d \times [0, \infty]$ and has a spectral gap Δ (in particular it is an insulator, but not a strong topological insulator). The perturbations $H = H_0 + V$ with spectral gap in Δ and $V \in \mathcal{A}$ can therefore be thought of as (possibly topological) insulators realized on the surface of a substrate described by H_0.

If V acts only on a finite slab $\mathbb{Z}^d \times [0, L]$ and commutes with H_0 then one just has a d-dimensional topological insulator on top of H_0 without interaction, hence one can eliminate the substrate and go over to a purely d-dimensional description. One can then compute the surface topological invariants of H from the surface part of its Fermi projection. However, this is not a stable procedure, since any generic perturbation will couple V with the substrate, therefore one cannot split the Fermi projection as a direct sum of a surface part in $\mathcal{A}$ and a substrate part in $M(\mathcal{A})$ anymore. In the relative formulation the surface topological invariants stay well-defined as the relative Chern numbers

$$\langle \mathrm{Ch}_{\mathcal{T}\otimes\mathrm{Tr},\alpha}, [e_F]_0^M - [e_0]_0^M \rangle$$

for α as in Theorem 4.3.19 where one compares the Fermi projection of H with that of the undisturbed substrate described by H_0. This shows that they actually are stable unperturbatively, i.e. invariant under homotopies that do not close the gap of the combined $(d+1)$-dimensional system. Moreover they are still classified by $K_i(\mathcal{A})$, no matter what exactly the substrate is.

Likewise, the bulk-boundary correspondence (see the later sections) can be pushed through if one e.g. realizes an interface between different topological phases on top of the same substrate.

4.3.6 Examples: Continuous models

In this section we will consider examples affiliated to the observable algebra $\mathcal{A} = M_N(C(\mathbb{R}^d_{0,\Omega}))$, i.e. matrix-valued differential operators with potentials but no magnetic field.

The easiest example are Schrödinger-type Hamiltonians

$$H = -\nabla^2 + V = -\sum_{i=1}^{d} \nabla_i^2 + V$$

with some random or periodic potential that leads to a spectral gap in some interval Δ. As a reference Hamiltonian one has the canonical choice $H_0 = -\nabla^2 + m$ for a constant m which is larger than $\sup \Delta$. Since the Fermi projection of H_0 then vanishes for the common gap Δ, the relative formulation of the gapped topological invariants immediately drops to an absolute one, where no comparison is required and the Fermi projection of H is naturally a matrix with elements in $\mathcal{A}$.

Being bounded from below this example Schrödinger-type Hamiltonians allow no chiral symmetry; for that examples are most easily found in the class of Dirac-type operators, e.g. the one-dimensional Hamiltonian

$$H_m = \begin{pmatrix} 0 & \nabla_x + m \\ -\nabla_x + m & 0 \end{pmatrix}$$

with spectral gap in $(-m, m)$, as one verifies by Fourier transforming to a matrix-valued multiplication operator. Here the bounded transform $F(H_m)$ and thus Fermi projection are only in the multiplier algebra $M(\mathcal{A})$, thus one needs to choose a reference Hamiltonian. More precisely, H_m is not strongly affiliated since its bounded transform is the function

$$x \in \mathbb{R} \mapsto \frac{1}{\sqrt{m^2 + x^2}} \begin{pmatrix} 0 & \imath x + m \\ -\imath x - m & 0 \end{pmatrix}$$

which is not a matrix over the one-point compactification $C(\mathbb{R})^{\sim} \simeq C(\mathbb{R} \cup \infty)$ but just over the bounded functions $C_b(\mathbb{R})$.

Due to the symmetry there is no natural choice for a reference Hamiltonian, but for any $m \neq 0$ it makes sense to look at pairs (H_m, H_{-m}) of which either can be considered as the reference and which leads to a well-defined K-theory class $[u_m u_{-m}^*]_1 \in K_1(\mathcal{A})$ where $u_m \in M(\mathcal{A})$ is the phase of the off-diagonal component of H_m. For opposite masses this unitary has winding number

$$\langle \mathrm{Ch}_{\mathcal{T},\theta}, [u_m]_1^M - [u_{-m}]_1^M \rangle = \mathrm{sgn}(m).$$

Those one-dimensional Dirac-type Hamiltonians are well-known for their robust zero-energy bound states that appear in domain wall configurations with opposite masses at $\pm\infty$ [68]. That is an expression of bulk-interface correspondence and follows from the non-vanishing relative Chern number (see Section 5.4 below).

A similar example in two dimensions is the massive Dirac Hamiltonian

$$H_m = \begin{pmatrix} m & \nabla_x + \imath\nabla_y \\ -\nabla_x + \imath\nabla_y & -m \end{pmatrix} \tag{4.3.8}$$

also with spectral gap in $(-m, m)$. Here one also has a gapped topological invariant if one compares two Hamiltonians of opposite mass, it is the Chern number of the class

$$[\chi(H_m < 0)]_0^M - [\chi(H_{-m} < 0)]_0^M \in K_0(\mathcal{A})$$

and can be computed using either the multiplier version of the Chern cocycle from Section 3.4 or by resolving the multiplier picture using a suspension into $K_1(S\mathcal{A})$. Using either way one obtains a relative Chern number of ± 1. In the case here one can also show that the Chern number can be computed by regularization

$$\begin{aligned} &\langle \mathrm{Ch}_{\mathcal{T},\theta}, [\chi(H_m < 0)]_0^M - [\chi(H_{-m} < 0)]_0^M \rangle \\ &= c \lim_{R\to\infty} \mathcal{T}\left(\chi_R e_+[[X, e_+], [Y, e_+]] - \chi_R e_-[[X, e_-], [Y, e_-]]\right) \end{aligned}$$

where $\chi_R \in L^\infty(\mathcal{A}) \simeq L^\infty(\mathbb{R}^2)$ is the characteristic function of the ball B_R and X, Y the two position operators (i.e. the momentum-space derivatives after Fourier transform). This can be derived as a consequence of Corollary 3.4.6 since one can mollify the projections such that the difference $e_+ - e_-$ becomes compactly supported and ends up with the above expression in a limit where the regularization is removed. The two terms in the sum evaluated individually converge to $\pm\frac{1}{2}$, hence one sometimes says in physics that the Hall conductance of a massive Dirac-Fermion is $\frac{1}{2}$ (for a discussion of this point see e.g. [19, 8.2.3]). Nevertheless, the value $\frac{1}{2}$ is not expected to be stable under perturbation (in fact the integral generically becomes ill-defined except for very special perturbations).

One can obviously construct such massive Dirac-type Hamiltonians from Clifford algebras in any dimension and they are chirally symmetric in odd dimensions. They always give examples of a non-vanishing top Chern number.

Another useful source of examples is a class of quadratic Hamiltonians

$$H = \sigma\nabla^2 + D + V$$

with σ a self-adjoint matrix with $\sigma^2 = \mathbb{1}$, D a first-order symmetric matrix-differential operator and a potential $V \in M(\mathcal{A})$. Such a quadratic Hamiltonian is strongly p-smooth for any $p \in (\frac{d}{2}, \infty]$ since it is iteratively built by modifying the Laplacian as sketched under Proposition 4.3.22. One can still use the reference Hamiltonian $H_0 = \sigma(\nabla^2 - m)$, where one chooses m large compared to Δ.

The most prominent member of this family is the regularized Dirac-Hamiltonian

$$H = \begin{pmatrix} \epsilon\nabla^2 + m & \nabla_x + \imath\nabla_y \\ -\nabla_x + \imath\nabla_y & -(\epsilon\nabla^2 + m) \end{pmatrix} \tag{4.3.9}$$

whose Fermi projection has the Chern number $\frac{1}{2}(\mathrm{sgn}(\epsilon) + \mathrm{sgn}(m))$ [125].

Another much-studied example arises from a Hamiltonian formulation of the shallow-water equations and has recently become relevant as a counter-example to some formulations of bulk-boundary correspondence [124, 125, 59, 126]. It is given by the two-dimensional differential operator

$$H = \begin{pmatrix} 0 & -\imath\nabla_x & -\imath\nabla_y \\ -\imath\nabla_x & 0 & -\imath(f - \epsilon\nabla^2) \\ -\imath\nabla_y & \imath(f - \epsilon\nabla^2) & 0 \end{pmatrix} \tag{4.3.10}$$

with some constants $f, \epsilon \neq 0$. It is strongly affiliated to $\mathcal{A} = C(\mathbb{R}^d_{0,*})$ which is easily verified by an explicit computation in momentum space (indeed making the Chern numbers of the bands well-defined was part of the motivation for the introduction of the second order term in [124] which physically represents an odd viscosity). The spectrum consists of three bands, one infinitely degenerate flat band with energy 0 and two bands with dispersion relation

$$E_\pm(k) = \pm\sqrt{k^2 + (f + \epsilon k^2)^2}.$$

The central band has Chern number 0 and the other bands have Chern number $\pm(\mathrm{sgn}(\epsilon) + \mathrm{sgn}(f))$ respectively. As one can conclude from the flat band this Hamiltonian is somewhat of an edge case of our formalism since it is strongly affiliated but not resolvent-affiliated; an explicit computation shows that $(H + \imath)^{-1}$

is a matrix in $M_3(\mathcal{A}^\sim) \setminus M_3(\mathcal{A})$. For that reason the strong affiliation need not be stable and indeed adding a general first-order differential operator will perturb the bounded transform into the multiplier algebra. There is, however, a limited stability under perturbations $V \in M_3(\mathcal{A})$ whose upper left entry vanishes since most matrix entries of the resolvent $(H+\imath)^{-1}$ are in $\mathcal{A}$. This includes in particular all perturbations which preserve the off-diagonal shape of H.

One can also study the model without the odd viscosity term, i.e. for $\epsilon = 0$, which then requires a reference Hamiltonian. The obvious choice is to consider a pair (H_f, H_{-f}) with opposite values of the Coriolis parameter and which then has a relative Chern number of value ± 2.

4.4 Gapless topological invariants

The previous section was concerned with topological invariants that could be associated to spectral gaps of (pairs of) Hamiltonians. In contrast, this section will abstractly discuss topological invariants associated to spectral regions Δ that *obstruct* the formation of a gap in Δ. Most prominently this concerns the boundary topological invariants that can be associated to topological insulators with a bulk gap, but there are also other examples e.g. Dirac Hamiltonians which carry topological charges.

One of the takeaways from the theory of gapped topological invariants was the fact that they can also be well-defined without a true spectral gap in the presence of a mobility or pseudo-gap. For the gapless invariants one can analogously derive no-go results in the same vein: Their non-triviality may not only prevent the formation of spectral gaps but also that of mobility gaps or (too strict) pseudogaps. This generalizes existing results which show that the boundary states associated to half-space restrictions of topological insulators cannot be dynamically localized [103][29][111].

Let $\mathcal{A}$ again be a C^*-algebra and for the numerical invariants we assume that it is part of a tracial dynamical system $(\mathcal{A}, \theta, \mathcal{T})$. A Hamiltonian H is again a self-adjoint $\mathcal{A}$-multiplier, however, instead of a spectral gap we require to have invertibility up to $\mathcal{A}$:

Definition 4.4.1 *Let H be a self-adjoint $\mathcal{A}$-multiplier of an observable algebra $\mathcal{A}$ with $\Delta \subset \mathbb{R}$ a (possibly infinite) spectral interval. We say that Δ is an $\mathcal{A}$-essential gap if $f(H) \in \mathcal{A}$ for each $f \in C_c(\Delta)$.*

For bounded H this is equivalent to the $\Delta \cap \sigma(H + \mathcal{A}) = \emptyset$ with the image in the Corona algebra $M(\mathcal{A})/\mathcal{A}$. For unbounded H a typical way to assert the existence of an $\mathcal{A}$-essential gap is to find a symmetric $v \in M(\mathcal{A})$ such that $(H - E)^2 + v$ is strictly positive for $E \in \Delta$ and $v(1 + H^2)^{-1} \in \mathcal{A}$. In fact, if such a v exists at all, one can always use $v = f(H) \in \mathcal{A}$ for a non-negative function f which is compactly supported within the essential gap Δ.

Definition 4.4.2 *Let H be a self-adjoint Hamiltonian with $\mathcal{A}$-essential gap Δ. Let $\chi_\Delta \in C^\infty(\mathbb{R})$ a smooth function with $\chi_\Delta(t) = -1$ below Δ and $\chi_\Delta(t) = 1$ above Δ. We associate to H the class $[u_\Delta]_1 \in K_1(\mathcal{A})$ represented by*

$$u_\Delta = e^{\pi\iota(\chi_\Delta(H)+1)} \in M_N(\mathcal{A}^\sim) .$$

If H anticommutes with the chiral symmetry J and $0 \in \Delta$ then choose $0 \in \Delta = -\Delta$ and χ_Δ to be anti-symmetric. We then associate to H the class $[e_\Delta]_0 - [s(e_\Delta)]_0 \in K_0(\mathcal{A})$ represented by

$$e_\Delta = e^{-\iota\frac{\pi}{2}\chi_\Delta(H)} \begin{pmatrix} \mathbb{1}_{\frac{N}{2}} & 0 \\ 0 & 0_{\frac{N}{2}} \end{pmatrix} e^{\iota\frac{\pi}{2}\chi_\Delta(H)} \in M_N(\mathcal{A}^\sim) .$$

For any restriction α of θ to a $\mathbb{R}^n$-action we associate to H the numerical gapless invariants

$$\langle Ch_{\mathcal{T},\alpha}, [x_\Delta]_i \rangle \in \mathbb{R}$$

as a pairing with $K_i(\mathcal{A})$, $i = n \mod 2$.

If H is bounded then those K-theory classes are by construction just images of the Fermi projection respectively Fermi unitary of $H \mod \mathcal{A}$ under the K-theoretical boundary maps of the exact sequence

$$0 \to \mathcal{A} \to M(\mathcal{A}) \to M(\mathcal{A})/\mathcal{A} \to 0.$$

It is therefore easy to see that the K-theory classes are well-defined and independent of the switch function, though the latter can be easily seen directly via homotopy. One could also extend the range of applicability by defining the gapless invariants for pairs of Hamiltonians (H, H_0) which have an interval Δ with $f(H) - f(H_0) \in \mathcal{A}$ for all $f \in C_c(\Delta)$, but this will not be pursued further here.

One obtains numerical Chern numbers by pairing with appropriate Chern cocycles and does not need to worry about smoothness or summability conditions as one can always find a regular enough representative by spectral invariance of the domain $\mathcal{A}_{\mathcal{T},\theta}$.

For the pairings with the Chern cocycles one can easily formulate index theorems using Theorem 2.3.5. Those numerical invariants are also stable and stay well-defined even under nice enough perturbations from larger subalgebras of $L^\infty(\mathcal{A})$ than $M(\mathcal{A})$. Let us also mention another elegant way to obtain an index theorem for the gapless invariants, which involves H directly without any functional calculus, thereby highlighting the naturalness of the construction:

Theorem 4.4.3 *Let $(\mathcal{A}, \theta, \mathcal{T})$ be tracial dynamical system and assume that θ is an even-dimensional action. As in Chapter 2 one construct the von Neumann algebra $\mathcal{N} = L^\infty(\mathcal{A} \rtimes_\theta \mathbb{R}^d)$ with trace $\hat{\mathcal{T}}_\xi$ and the canonical Dirac operator $\mathbf{D}$ from Chapter 2. If H is a θ-smooth $\mathcal{A}$-multiplier with $\mathcal{A}$-essential gap Δ containing 0 then the operator $\kappa\mathbf{D} + \imath H$ is affiliated to $\mathcal{N}$ and $\hat{\mathcal{T}}_\xi$-Fredholm for small enough κ.*

For u_Δ as in Definition 4.4.2 one then has

$$\langle Ch_{\mathcal{T},\theta}, [u_\Delta]_1\rangle = \lim_{\kappa\to 0} \hat{\mathcal{T}}_\xi\text{−Ind}(\kappa D + \imath H).$$

Proof. The subalgebra of smooth elements $\mathcal{A}_\theta$ of a $\mathcal{A}$ together with the semifinite von Neumann algebra $\mathcal{N}$ and the Dirac operator $\mathbf{D}$ constitute a so-called semifinite spectral triple ([35, 112] for definitions). Since H is strictly smooth and invertible modulo $\mathcal{A}$ it is an unbounded Callias potential in the sense of [112] and the present statement is exactly the main result of loc.cit. applied to the current situation. □

A similar index theorem exists for the even classes as well. If H is resolvent-affiliated to $\mathcal{A}$ the Callias operator $\kappa\mathbf{D} + \imath H$ (for any $\kappa > 0$) usually also represents the Kasparov product between unbounded cycles defined using from $\mathbf{D}$ and H (which is then independent of κ).

The most important examples for gapless invariants are the boundary invariants of topological systems for which there will be plenty in the following. Let us therefore look at an example for a gapless invariant in the bulk.

The massless Dirac Hamiltonian

$$H = \imath\sigma \cdot \nabla$$

for $\sigma = (\sigma_1, ..., \sigma_d)$ a representation of the complex Clifford algebra is an $\mathcal{A}$-multiplier of the algebra $\mathcal{A} = C(\mathbb{R}^d_{0,*})$. Its resolvent lies in $M_N(\mathcal{A})$, hence any interval Δ is an $\mathcal{A}$-essential gap for H. If d is even then H anticommutes with the chiral symmetry $J = \sigma_{d+1}$. Via Fourier transform H becomes a function $k \in \mathbb{R}^d \mapsto H_k = \sigma \cdot k \in M_N(C(\mathbb{R}^d))$ and one can compute the gapless invariants via the Chern cocycles, but this is rather hard to do explicitly since it involves smooth functional calculus. Since we are here in a situation of commutative geometry an alternate path is to use the Callias respectively Boutet de Monvel index formulas which give more convenient expressions, for example the odd invariant is [58]

$$\langle \mathrm{Ch}_{\mathcal{T},\theta}, [u_\Delta]_1 \rangle = \lim_{R\to\infty} c_d \int_{\partial B_R(0)} \mathrm{Tr}(Q(\mathrm{d}Q)^{\wedge(d-1)}) ,$$

with $Q = \mathrm{sgn}(H)$ and c_d a normalization constant. The even or odd top Chern number of $[u_\Delta]_1$ respectively $[e_\Delta]_0$ is equal to ± 1 [112] (depending on conventions for σ).

Coming back from the example, let us next discuss basic stability properties of the gapless invariants:

Proposition 4.4.4 *Let $t \in [0,1] \mapsto H_t$ be a gap-continuous map of Hamiltonians with common $\mathcal{A}$-essential gap Δ. Then the gapless invariants do not depend on t.*

Proof. By gap-continuous we mean that $t \mapsto f(H_t)$ is norm-continuous for any $f \in C_0(\mathbb{R})$, which implies that the representatives of the K-theory classes of the gapless invariants also change norm-continuously, hence the classes are constant. □

The question is therefore simply one of asserting that certain perturbations do preserve the $\mathcal{A}$-essential gap. The easiest criterion is

Proposition 4.4.5 *Assume that H has an $\mathcal{A}$-essential gap Δ and $V = V^* \in M(\mathcal{A})$ is such that $V(H+\imath)^{-1} \in \mathcal{A}$. Then Δ is also an $\mathcal{A}$-essential gap of $H+V$.*

Proof. For any function $f \in C_0(\Delta)$ one has $f(H) - f(H+V) \in \mathcal{A}$ and thus $f(H+V) \in \mathcal{A}$ which is one of the equivalent criteria given above. □

A special case is that of resolvent-affiliated H, i.e. if $(H+\imath)^{-1} \in \mathcal{A}$ then the gapless invariants are stable under arbitrary additive perturbations in $M(\mathcal{A})$. Since on the other hand one has at least $(H+\imath)^{-1} \in M(\mathcal{A})$ the gapless invariants are always stable under perturbations in $\mathcal{A}$ itself. Hence there is always a large class of perturbations that cannot open a spectral gap if the K-theory class is non-trivial.

We will now show that if one of the Chern numbers is non-trivial then it is also impossible to have mobility or too extreme pseudogaps. The strategy is to find a family of representatives whose Sobolev norm tends to 0 which therefore requires triviality. The argument presented here is a generalization of [111, Proposition 5.5.5] which proves delocalization of boundary states of topological insulators:

Proposition 4.4.6 *Assume that H is a smooth $\mathcal{A}$-multiplier and if $n \leq d$ is even then $H = -JHJ$ is chirally symmetric with Fermi level $E_F = 0$.*

(i) If H has a mobility gap then all Chern numbers must vanish for u_Δ respectively e_Δ.

(ii) If H has a pseudogap of order $\gamma > n$ anywhere in the $\mathcal{T}$-finite interval in Δ for odd n, or at 0 for even n then all weak Chern numbers with n generators must vanish.

Proof. Let $\alpha \subset \theta$ be an n-parameter action with odd n and let $E_0 \in \Delta$ be arbitrary. Write $\chi_\Delta = \chi(H \in \Delta)$ and set

$$u_{\Delta,\epsilon} \;=\; e^{2\pi\imath\, f_{\mathrm{Exp},\epsilon}(H)}\;,$$

for any $1 \geq \epsilon > 0$ and $f_{\mathrm{Exp},\epsilon} \in C^\infty(\mathbb{R})$ a choice of smooth functions which take the value 0 below E_0, 1 above E_0 and converge to $\chi(E > E_0)$ as $\epsilon \to 0$ for an arbitrary, but fixed $E_0 \in \Delta$. Then $\langle \mathrm{Ch}_{\mathcal{T},\alpha}, [e^{\imath\pi f_\epsilon(H)}]_1\rangle$ is independent of $\epsilon > 0$.

In the case (i), the mobility gap regime, $\left\|\nabla e^{\imath\pi f_\epsilon(H)}\right\|_{n+1}$ is bounded uniformly in ϵ and due to the regularity of the DOS one finds that $\left\|\mathbb{1} - e^{\imath\pi f_\epsilon(H)}\right\|_{n+1}$ is also uniformly bounded. Moreover, one has

$$\lim_{\epsilon\to 0}\left\|\mathbb{1} - e^{\imath\pi f_\epsilon(H)}\right\|_{n+1} = \lim_{\epsilon\to 0}\left\|\varphi(H)(\mathbb{1} - e^{\imath\pi f_\epsilon(H)})\right\|_{n+1} = 0$$

where we inserted a compactly supported function φ which is equal to 1 on $\bigcap_{1\geq\epsilon>0}\mathrm{supp}(f_\epsilon)$ and then used $\varphi(H) \in L^{n+1}(\mathcal{A})$ together with continuity of the trace, Lemma 1.3.1(ii). Continuity of the Chern cocycle implies

$$\langle \mathrm{Ch}_{\mathcal{T},\alpha}, [e^{\imath\pi f_\epsilon(H)}]_1\rangle = \langle \mathrm{Ch}_{\mathcal{T},\alpha}, [\mathbb{1}]_1\rangle = 0.$$

In the pseudogap regime of case (ii) we may assume that the pseudogap is at $E_0 = 0$ and instead bound

$$\left|\langle \mathrm{Ch}_{\mathcal{T},\alpha}, [e^{\imath\pi f_\epsilon(H)}]_1\rangle\right| \leq \left\|\mathbb{1} - e^{\imath\pi f_\epsilon(H)}\right\|_1 \left\|\nabla e^{\imath\pi f_\epsilon(H)}\right\|_\infty^n .$$

We can always choose f_ϵ in such a way that the compactly supported functions $g_\epsilon = 1 - e^{\imath\pi f_\epsilon}$ are just rescalings $g_\epsilon(\lambda) = g_1(\epsilon^{-1}F^{-1}(\lambda))$ for F^{-1} the inverse of the bounded transform and some suitable g_1. From the smooth functional calculus one gets

$$\nabla e^{\imath\pi f_\epsilon(H)} = \frac{1}{2\pi}\int_{\mathbb{C}} (\epsilon^{-1}F(H)+z)^{-1}(\epsilon^{-1}\nabla F(H))(\epsilon^{-1}F(H)+z)^{-1}\mathrm{d}z\wedge \mathrm{d}\overline{z}$$

and thus

$$\left\|\nabla e^{\imath\pi f_\epsilon(H)}\right\| \leq C_f \left\|\epsilon^{-1}\nabla F(H)\right\| .$$

Since $\mathbb{1} - e^{\imath\pi f_\epsilon(H)}$ is uniformly bounded in operator-norm and supported only on an interval of length $C\epsilon$ around 0 one further has from the density of states

$$\left\|\mathbb{1} - e^{\imath\pi f_\epsilon(H)}\right\|_1 \leq C\epsilon^\gamma .$$

Hence the pairing with the Chern cocycle must vanish if $\gamma > n$.

The same arguments apply almost verbatim in the odd case. □

There are some caveats here which are difficult to improve with this method: The first is that we assume that the density of states is at least Hölder-continuous. This is not always the case and there are are indeed examples where the even gapless invariant is non-trivial since H has an actual eigenvalue at 0 in the chiral case (this is generic for one-dimensional models, higher-dimensional examples can be found in Section 6.1). The second is that the mobility gap bound is assumed to hold uniformly, hence it could e.g. be possible in a chiral model that every interval not containing 0 is mobility gapped. Also there are weaker localization conditions such as the fractional moments bound which are not in obvious contradiction to the non-triviality of Chern numbers.

We observe that the massless Dirac-Hamiltonian in d dimensions H has a pseudogap: In the Fourier picture one simply has $|H_k| = |k|$, from which one concludes that $\mathcal{T}(\chi(|H| \in [-\epsilon,\epsilon]) = V_d\epsilon^d$ with V_d the volume of the d-dimensional unit ball. Thus there is a pseudogap of order $\gamma = d$, which is not only consistent with Proposition 4.4.6 it more strongly demonstrates that the derived lower bound

$\gamma > d$ is sharp. Comparing with Proposition 4.3.18 a gapped n-parameter Chern number becomes well-defined based on a pseudogap when $\gamma > n$, which is exactly when the corresponding gapless Chern number must become trivial.

About the massless Dirac-Hamiltonian we can now also say, since it is resolvent-affiliated to $C(\mathbb{R}^d_{0,\Omega})$, that the addition of a random or periodic potential from $M(C(\mathbb{R}^d_{0,\Omega}))$ (preserving the chiral symmetry in even dimension) will not change the gapless invariants. In odd dimensions such a random potential can in particular not lead to a mobility gap anywhere in the spectrum by Proposition 4.4.6. On the other hand, the even-dimensional Dirac-operators immediately become gapped if one adds the symmetry-breaking mass term $m\sigma_{d+1}$.

5 Algebraic bulk-boundary correspondence

A quantum system which is topologically non-trivial and gapped in the bulk generically exhibits robust surface states on its boundary. One can motivate this by noting that any continuous deformation of a topologically insulator to a trivial insulator in parameter space requires that the spectral gap closes along the path. Therefore the same should happen if one realizes such a continuous interface in real space which would result in gap-filling states. For example, a one-dimensional wire with chiral symmetry like the SSH-chain [117] will have 0-modes on its ends proportional to the winding number in the bulk. In two dimensions a Quantum Hall insulator has conducting chiral edge modes which fill the bulk gaps and whose conductivity is proportional to the bulk two-dimensional Chern number which is itself proportional to the 0-temperature Hall conductance [74]. For these two examples it is now understood that this is a consequence of K-theory since all of the above observable quantities are pairings with certain K-theory classes in a bulk or a surface algebra which are linked via the connecting maps. Let us refer to [103] for more historical context.

Let us also emphasize that this algebraic bulk-boundary correspondence requires numerous algebraic consistency conditions and even then not every non-trivial bulk invariant will result in surface states for every possible interface.

5.1 The strongly affiliated case

In this chapter we assume throughout that there is an exact sequence

$$0 \to \mathcal{E} \to \hat{\mathcal{A}} \to \mathcal{A} \to 0 \tag{5.1.1}$$

of separable C^*-algebras. While the algebraic statements are completely general, we will think of elements of $\mathcal{E}$ as edge-observables, $\mathcal{A}$ as bulk observables and of $\hat{\mathcal{A}}$ as an algebra of halfspace operators which consists of restrictions of bulk observables with boundary terms in $\mathcal{E}$. This terminology is used to give a proper physical interpretation and flavor. Nevertheless, $\hat{\mathcal{A}}$ could as well represent a system with a defect of higher codimension instead of a halfspace if $\mathcal{E}$ describes the observables localized around such a defect.

The method of algebraic bulk-boundary correspondence is to map K-theory classes over the bulk, i.e. in $K_i(\mathcal{A})$, to K-theory classes in $K_i(\mathcal{E})$ via the connecting

maps. While that is of course always possible, it only makes sense to do so if the images under the connecting maps can again be expressed in terms of representatives that are related to a physical system of interest. For example, if we have a self-adjoint $\mathcal{E}$-multiplier $\hat{H}$ as a Hamiltonian on a halfspace whose bulk limit is invertible in some spectral interval Δ then the natural goal would be to express the class of the gapless topological invariants $[u_\Delta]_1 \in K_1(\mathcal{E})$ in terms of a class in $K_0(\mathcal{A})$. This method was pioneered for two-dimensional tight-binding models in [74] and has since then been applied and extended in many works (e.g. [75, 79, 103, 25, 2, 53, 102]) with notable refinements coming from the use of KK-theory, twisted equivariant K-theory and real or van Daele K-theory. In this work we stick to the simplest setup of complex K-theory since the main point is the investigation of complications that arise from Hamiltonians that are (unbounded) multipliers and the results should generalize without too much difficulty.
We are also focused exclusively on the situation of invariants derived from one (or two) Hamiltonians as in Chapter 4. For completeness let us mention that some topological invariants arise differently, e.g. for a periodically driven Floquet system one constructs the bulk invariant from the propagator which is a unitary that should have a spectral gap [108], while in the context of the Levinson theorem in scattering theory they are constructed from wave operators [72]. The principle of bulk-boundary correspondence should always apply but one has to construct relevant images on a case-by-case basis.

To work with unbounded operators we need to clarify what a lift of an unbounded multiplier is supposed to be under a surjective homomorphism $q : \mathcal{A} \to \mathcal{B}$. For a bounded multiplier this is clear since there is a canonical extension $q : M(\mathcal{A}) \to M(\mathcal{B})$ defined by the property

$$q(\hat{m}) = m \quad \Leftrightarrow \quad q(\hat{m}a) = mq(a),\ q(a\hat{m}) = q(a)m, \quad \forall a \in \mathcal{A}.$$

In the unbounded case one retreats to bounded functions:

Definition 5.1.1 *Let $q : \mathcal{A} \to \mathcal{B}$ be a surjective homomorphism of C^*-algebras and denote the canonical extension $q : M(\mathcal{A}) \to M(\mathcal{B})$ by the same letter. If $\hat{H}$ is a self-adjoint $\mathcal{A}$-muliplier and H a self-adjoint $\mathcal{B}$-multiplier, then we say that $\hat{H}$ is a lift of H (under q) if one has any of the equivalent conditions*

(i) $q(F(\hat{H})) = F(H)$,

(ii) $q(f(\hat{H})) = f(H)$ *for each function* $f \in C_0(\mathbb{R})$.

(iii) $q((\hat{H}+\imath)^{-1}) = (H+\imath)^{-1}$,

(iv) $q(f(\hat{H})) = f(H)$ *for each bounded function* $f \in C_b(\mathbb{R})$.

To see that those conditions are equivalent one notes that $(i) \implies (ii)$, $(ii) \iff (iii)$ and $(iv) \implies (i)$ are trivial and $(ii) \implies (iv)$ holds since the density of $(1+\hat{H}^2)^{-\frac{1}{2}}\mathcal{A}$ implies that one only needs to check that

$$q(f(\hat{H})(1+\hat{H}^2)^{-\frac{1}{2}}a) = f(H)(1+H^2)^{-\frac{1}{2}}q(a)$$

and its conjugate hold for all $a \in \mathcal{A}$.

An alternative approach would be to immediately replace any unbounded operator by its bounded transform, which is consistent with the notion of lift chosen above. As we will see, that would not always be wise to do, since e.g. different self-adjoint extensions of the same symmetric multiplier are most easily compared via resolvent-differences and those are difficult to express in terms of bounded transforms.

Let us first describe the best-understood setting of strongly affiliated Hamiltonians. Recall that this is the generic case for unital algebras of tight-binding models such as $C(\mathbb{T}^d_{\mathbf{B},\Omega})$ or resolvent-affiliated Hamiltonians that are bounded from below. In that case one has a straightforward generalization of the standard theory [103, Proposition 4.3.1-2]:

Proposition 5.1.2 *Let H be self-adjoint $M_N(\mathcal{A})$-multiplier with a spectral gap Δ and let with $E_F \in \Delta$. Further let $\hat{H}$ be a self-adjoint $M_N(\hat{\mathcal{A}})$-multiplier that lifts H in the exact sequence (5.1.1)*

Assume strong affiliation, i.e.

$$F(H) \in M_N(\mathcal{A}^\sim), \qquad \text{and} \qquad F(\hat{H}) \in M_N(\hat{\mathcal{A}}^\sim) \tag{5.1.2}$$

then

(i) The exponential map $\operatorname{Exp}: K_0(\mathcal{A}) \to K_1(\mathcal{E})$ *maps* $[e_F]_0$ *to* $[\hat{u}_\Delta]_1$ *where*

$$\hat{u}_\Delta \;=\; e^{\pi\imath(\chi_\Delta(\hat{H})+1)} \;\in\; M_N(\mathcal{E}^\sim)\,.$$

(ii) *If the CH holds for both H and $\hat{H}$, the index map* $\mathrm{Ind} : K_1(\mathcal{A}) \to K_0(\mathcal{E})$ *maps* $[u_F]_1$ *to* $[\hat{e}_\Delta]_0 - [0_{\frac{N}{2}} \oplus \mathbb{1}_{\frac{N}{2}}]_0$ *where*

$$\hat{e}_\Delta = e^{-\imath \frac{\pi}{2} \chi_\Delta(\hat{H})} \begin{pmatrix} \mathbb{1}_{\frac{N}{2}} & 0 \\ 0 & 0_{\frac{N}{2}} \end{pmatrix} e^{\imath \frac{\pi}{2} \chi_\Delta(\hat{H})} \in M_N(\mathcal{E}^\sim) \ .$$

Here the function χ_Δ is chosen exactly as in Definition 4.4.2.

Proof. The condition (5.1.2) implies $e_F \in \mathcal{A}^\sim$, which is required to define a class in $K_0(\mathcal{A})$ in the first place and the second condition implies that $\frac{1}{2}(1 - \chi_\Delta(\hat{H}))$ is a lift of e_F to a contraction in $M_N(\hat{\mathcal{A}}^\sim)$. The construction of the images is then standard and follows exactly as in [103]. □

If $\mathcal{A}$ and $\hat{\mathcal{A}}$ are already unital then (5.1.2) is trivial. This concerns mostly the case of tight-binding models with a semisplit bulk-boundary exact sequence, i.e. the lifting condition is equivalent to the condition that $\hat{H}$ can be written in the form $\hat{H} = PHP + \hat{v}$ with P the projection to the positive half-space (which shall then be the unit of $\hat{\mathcal{A}}$).

In the non-unital case with unbounded Hamiltonians there are many more possibilities. Let us consider the bulk algebra $\mathcal{A} = C_0(\mathbb{R}^d)$ as appropriate for translation-invariant models in a continuous space. We already introduced the massive Dirac Hamiltonian (4.3.8) whose Fermi projection does not even define an element of $K_0(\mathcal{A})$ on its own. Since we need a reference Hamiltonian in the bulk, there can at most be a well-defined relative bulk-boundary correspondence which also compares two systems with boundaries. But even assuming strong affiliation in the bulk we are not done, since one bulk Hamiltonian can lift to many different halfspace Hamiltonians and it can be difficult to identify the strongly affiliated lifts (if they even exist). The simplest Hamiltonian one can write down is the Laplacian $H = -\nabla^2$, and restricting it to $C_0^\infty(\mathbb{R}^{d-1} \times \mathbb{R}_+)$ one obtains a symmetric operator $\hat{H}$. As we will see below, an exact sequence for such a halfspace problem can be obtained from a smooth Toeplitz extension. For $d > 1$ it is well-known that there are infinitely many self-adjoint extensions of $\hat{H}$ corresponding to different boundary conditions. Many of those do not even lead to multipliers of the half-space algebra $\hat{\mathcal{A}}$, for example if we require translation-invariance in the directions parallel to the boundary (as we will do in the examples below) or due to certain continuity conditions. However, even among those extensions which do define multipliers only a small subset of can satisfy the condition (5.1.2). Staying in the example of the Laplacian, we do know (and will see in Section 6.3.2) that at least the halfspace-Laplacian with Dirichlet boundaries is strongly affiliated to an appropriate algebra. Let us emphasize that strong affiliation is a rather tricky condition to verify: The bounded transform of a halfspace differential operator is very difficult to compute explicitly, therefore one would want to rely on general principles. The self-adjoint extensions are conventionally classified using the von Neumann theory, which basically describes how the formal Cayley transform of a symmetric multiplier must be extended to obtain a unitary operator. Thus one can efficiently compare different extensions via the differences of their resolvents. Unfortunately, the bounded transform of a fixed self-adjoint multiplier does not depend norm-continuously on the resolvent in general, hence there is no clear path to single out those extensions (respectively boundary conditions) that lead to strongly affiliated self-adjoint operators. If the extensions are bounded from below the problem sometimes reduces to the question of resolvent-affiliation which we will find to be a much more tractable problem.

5.2 Relative bulk-boundary correspondence

As was argued above, when either the bulk or the halfspace Hamiltonian fails to be strongly affiliated one has to resort to a relative bulk-boundary correspondence which compares two different but similar Hamiltonians. In the multiplier picture the modification of Proposition 5.1.2 is straightforward:

Proposition 5.2.1 *Let H be a self-adjoint $\mathcal{A}$-multiplier with reference Hamiltonian H_0 such that $F(H) - F(H_0) \in \mathcal{A}$ and both have the common spectral gap Δ. If $(\hat{H}, \hat{H}_0)$ are lifts of (H, H_0) such that the pair of bounded transforms $(F(\hat{H}), F(\hat{H}_0))$ lies in $\mathbb{P}(M(\hat{\mathcal{A}}), \hat{\mathcal{A}})$ then the image in $K_1(\mathcal{E})$ under the exponential map is given by*

$$[e^{\pi\iota(\chi_\Delta(\hat{H})+1)}]_1^M - [e^{\pi\iota(\chi_\Delta(\hat{H}_0)+1)}]_1^M = \operatorname{Exp}\left([\chi(H < \Delta)]_0^M - [\chi(H_0 < \Delta)]_0^M\right).$$

If $H, \hat{H}, H_0, \hat{H}$ have a chiral symmetry then

$$\begin{aligned}&[e^{-\iota\frac{\pi}{2}\chi_\Delta(\hat{H})}\begin{pmatrix}\mathbb{1}_{\frac{N}{2}} & 0\\ 0 & 0_{\frac{N}{2}}\end{pmatrix}e^{\iota\frac{\pi}{2}\chi_\Delta(\hat{H})}]_0^M - [e^{-\iota\frac{\pi}{2}\chi_\Delta(\hat{H}_0)}\begin{pmatrix}\mathbb{1} & 0\\ 0 & 0\end{pmatrix}e^{\iota\frac{\pi}{2}\chi_\Delta(\hat{H}_0)}]_0^M\\ &= \operatorname{Ind}\left([u_F]_1^M - [u_0]_1^M\right) \in K_0(\mathcal{E})\end{aligned}$$

with the Fermi unitaries as in Definition 4.3.4.

Proof. Since χ_Δ is a C_0-function of the bounded transform the contraction $\hat{e} = \frac{1}{2}(\mathbb{1} - (\chi_\Delta(\hat{H}), \chi_\Delta(\hat{H}_0)))$ lies in $\mathbb{P}(M(\hat{\mathcal{A}}), \hat{\mathcal{A}})$ and is a lift of the bulk Fermi projections $e := (\chi(H < \Delta), \chi(H_0 < \Delta)) \in \mathbb{P}(M(\mathcal{A}), \mathcal{A})$, hence the image under the exponential map is represented by the unitary

$$\left(e^{\pi\imath(\chi_\Delta(\hat{H})+1)}, e^{\pi\imath(\chi_\Delta(\hat{H}_0)+1)}\right) \in \mathbb{P}(\mathrm{Ker}(\overline{q}), \mathcal{E})$$

with $\overline{q} : M(\hat{\mathcal{A}}) \to M(\mathcal{A})$.

The proof is then completed by using the expressions for the connecting maps from Proposition 1.5.4. Let us also clarify that in our notation it is understood that

$$\begin{aligned}&[e^{\pi\imath(\chi_\Delta(\hat{H})+1)}]_1^M - [e^{\pi\imath(\chi_\Delta(\hat{H}_0)+1)}]_1^M \\ &= [\mathbb{1} \otimes (\mathbb{1} - f) + e^{\pi\imath(\chi_\Delta(\hat{H})+1)} \otimes f]_1^M - [\mathbb{1} \otimes (\mathbb{1} - f) + e^{\pi\imath(\chi_\Delta(\hat{H}_0)+1)} \otimes f]_1^M\end{aligned}$$

where $f \in \mathbb{K}$ is a rank-one-projection which is fixed once and for all.

□

If all multipliers are bounded then one can also replace $F(H)$ at every occurrence with H itself. Since the conditions look natural enough one is tempted to think that relative bulk-boundary correspondence holds in practically any relevant situation. That impression is deceptive, since the condition

$$F(\hat{H}) - F(\hat{H}_0) \in M_N(\hat{\mathcal{A}}) \tag{5.2.1}$$

is rather subtle and will fail in many situations. To see what can go wrong, let us think of H_0 as a differential operator with $H = H_0 + V$ where $V \in M(\mathcal{A})$ is a potential. If we choose a lift $\hat{H}_0$ of H_0 then it is natural to look for lifts $\hat{H}$ satisfying (5.2.1) via the Ansatz $\hat{H} = \hat{H}_0 + \hat{V}$ with $\hat{V} \in M(\hat{\mathcal{A}})$. The problem is that usually the only general statement one can make about the bounded transforms is that $F(\hat{H}) - F(\hat{H}_0)$ lies in the same C^*-algebra as $\hat{V}(\hat{H}_0 + \imath)^{-1}$. Since $\hat{V}$ being only a multiplier of $\hat{\mathcal{A}}$ cannot be avoided, one can typically enforce (5.2.1) only by assuming that $\hat{H}, \hat{H}_0$ are resolvent-affiliated to $\hat{\mathcal{A}}$.

This resolvent-affiliated case is so important that it is worth to specialize Proposition 5.2.1 to that case:

Corollary 5.2.2 *Let H, H_0 be resolvent-affiliated self-adjoint $\mathcal{A}$-multipliers with $H - H_0 \in M(\mathcal{A})$ and common spectral gap Δ. If $\hat{H}$, $\hat{H}_0$ are resolvent-affiliated $\hat{\mathcal{A}}$-multipliers with $\hat{H} - \hat{H}_0 \in M(\hat{\mathcal{A}})$ and lifts of H, H_0 respectively then*

$$[e^{\pi\iota(\chi_\Delta(\hat{H})+1)}]_1 - [e^{\pi\iota(\chi_\Delta(\hat{H}_0)+1)}]_1 = \mathrm{Exp}\left([\chi(H<\Delta)]_0^M - [\chi(H_0<\Delta)]_0^M\right).$$

If $H, H_0, \hat{H}, \hat{H}_0$ have chiral symmetry then

$$\begin{aligned}&[e^{-\iota\frac{\pi}{2}\chi_\Delta(\hat{H})}\begin{pmatrix}\mathbb{1} & 0\\ 0 & 0\end{pmatrix}e^{\iota\frac{\pi}{2}\chi_\Delta(\hat{H})}]_0 - [e^{-\iota\frac{\pi}{2}\chi_\Delta(\hat{H}_0)}\begin{pmatrix}\mathbb{1} & 0\\ 0 & 0\end{pmatrix}e^{\iota\frac{\pi}{2}\chi_\Delta(\hat{H}_0)}]_0\\ &= \mathrm{Ind}\left([u_F]_1^M - [u_0]_1^M\right) \in K_0(\mathcal{E}).\end{aligned}$$

Proof. Since the differences are bounded multipliers Proposition A.5 implies $F(H) - F(H_0) \in \mathcal{A}$ and $F(\hat{H}) - F(\hat{H}_0) \in \hat{\mathcal{A}}$.

Note now that classes on the left-hand sides are no longer in the multiplier picture but the actual difference of classes in $K_i(\mathcal{E})$, since the representatives lie in $\mathcal{E}^\sim$ due to the bulk gap and resolvent affiliation. □

As suggested above, the way to apply this result is to determine at first a lift $\hat{H}_0$ of H_0 which is resolvent-affiliated, then one obtains a lift of $\hat{H}$ by adding to that any lift of $H - H_0 \in M(\mathcal{A})$. In particular for unbounded $\hat{H}_0$ one should recall that by Kato-Rellich $\hat{H}$ and $\hat{H}_0$ have equal domains, hence this result allows us to e.g. compare any two similar enough bulk Hamiltonians which are restricted to some region with the same boundary conditions. Notably, the resolvent-affiliation makes the relative bulk-boundary correspondence very stable under bounded perturbations, therefore it is a very desirable property to have.

In applications one will want to construct the lift $\hat{H}_0$ by starting with a symmetric $\hat{\mathcal{A}}$-multiplier and then attempt to construct self-adjoint extensions. For example, if H_0 is a differential operator it is natural to restrict it to a formally self-adjoint expression on a half-space. The immediate question is, assuming H_0 is already resolvent-affiliated in the bulk, which boundary conditions does one need to impose such that the corresponding self-adjoint extension is resolvent-affiliated to the half-space algebra $\hat{\mathcal{A}}$? Obviously, we can say very little about that on an abstract level, however, we can exhibit the general structure of the set of all resolvent-affiliated extensions of a symmetric halfspace multiplier.

For that we first need to recall some details on self-adjoint extensions in the Hilbert-module setting from [132]:

Theorem 5.2.3 *Let $\hat{H}$ be a symmetric $\hat{\mathcal{A}}$-multiplier and $\hat{H}_u$ an extension of $\hat{H}$ that is a self-adjoint $\hat{\mathcal{A}}$-multiplier. Then the Cayley transform*

$$\mathcal{C}(\hat{H}_u) = (\hat{H}_u - \imath)(\hat{H}_u + \imath)^{-1}$$

of $\hat{H}_u$ takes the form

$$\mathcal{C}(\hat{H}_u) = \mathcal{C}(\hat{H}) + u$$

where $u \in M(\hat{\mathcal{A}})$ is a partial isometry and the formal Cayley-transform $\mathcal{C}(\hat{H}) \in M(\hat{\mathcal{A}})$ is a partial isometry with initial subspace $\mathbb{1} - u^*u$ and final subspace $\mathbb{1} - uu^*$. The projections $e_- = u^*u$ and $e_+ = uu^*$ are the projections to the deficiency subspaces $\mathrm{Ran}(e_\pm) = \mathrm{Ker}(\hat{H}^* \mp \imath)$. The correspondence between such partial isometries u and extensions $\hat{H}_u$ is one-to-one with the domain of the extensions

$$\mathrm{Dom}(\hat{H}_u) = \{a + a_+ + ua_+ : a \in \mathrm{Dom}(\mathcal{A}), a_+ \in \mathrm{Ker}(\hat{H}^* - \imath)\}.$$

If $\hat{H}$ is represented on a Hilbert space, the domains of the extensions can be constructed using the analogous formula. Notably, the extensions that are again $\hat{\mathcal{A}}$-multipliers are precisely those for which the partial isometry u is a multiplier.

We often want to impose resolvent-affiliation to $\hat{\mathcal{A}}$, which turns out to be a rather strict restriction on the allowed extensions:

Corollary 5.2.4 *Let $\hat{H}_u$ be a self-adjoint extension of a symmetric $\hat{\mathcal{A}}$-multiplier $\hat{H}$ and a lift of the self-adjoint $\mathcal{A}$-multiplier H.*

If $\hat{H}_u$ is resolvent-affiliated to $\hat{\mathcal{A}}$ and $\hat{H}_v$ any other self-adjoint extension of $\hat{H}$ then the following are equivalent:

(i) $\hat{H}_v$ *is resolvent-affiliated to* $\hat{\mathcal{A}}$ *and a lift of* H.

(ii) $f(\hat{H}_u) - f(\hat{H}_v) \in \mathcal{E}$ *for all* $f \in C_0(\mathbb{R})$.

(iii) $\mathcal{C}(\hat{H}_u) - \mathcal{C}(\hat{H}_v) \in \mathcal{E}$.

(iv) $u - v \in \mathcal{E}$.

Proof. We have $(i) \implies (ii)$ since $f(\hat{H}_u) - f(\hat{H}_v) \in \hat{\mathcal{A}}$ and $q(f(\hat{H}_u) - f(\hat{H}_v)) = f(H) - f(H) = 0$, hence $f(\hat{H}_u) - f(\hat{H}_v) \in \mathrm{Ker}(q) \cap \hat{\mathcal{A}} = \mathcal{E}$. By definition of u, v one has $(iii) \iff (iv)$ and the implications $(ii) \implies (iii)$ and $(iv) \implies (i)$ follow from

$$u - v = \mathcal{C}(\hat{H}_u) - \mathcal{C}(\hat{H}_v) = -2\imath\left((\hat{H}_u + \imath)^{-1} - (\hat{H}_v + \imath)^{-1}\right).$$

□

Hence, given any resolvent-affiliated extension $\hat{H}_u$ there is a one-to-one correspondence between resolvent-affiliated extensions $\hat{H}_v$ and partial isometries $u \in M(\hat{\mathcal{A}})$ satisfying $vv^* = uu^*$, $v^*v = u^*u$ and $u - v \in \mathcal{E}$. If we can guess (possibly on grounds of physical reasoning) or compute any extension $\hat{H}_u$ for which resolvent-affiliation holds then this one-to-one relationship can in principle be used to determine the complete set of resolvent-affiliated extension from the deficiency subspaces. In practice that can of course a task of formidable complexity. Note also that for differential operators the isometries u are in one-to-one correspondence with boundary conditions, however, the exact relation can be rather complicated and it is difficult to guess which changes in boundary condition are small enough to preserve resolvent-affiliation. We will compute an explicit example later in Section 5.3.2.

The next question is, can one also relate two halfspace Hamiltonians which are actually *equal* in the bulk, but differ by the choice of boundary conditions? In general, two different boundary conditions can give completely different boundary invariants or (in less favorable circumstances) lead to ill-defined boundary invariants, since there is usually enough freedom in the choice of self-adjoint extension to violate all affiliation conditions. However, if we stay in the class of resolvent-affiliated extensions there is a simple topological relationship:

Proposition 5.2.5 *Let $\hat{H}_u$, $\hat{H}_v$ be self-adjoint resolvent-affiliated extensions of a common symmetric $\hat{\mathcal{A}}$-multiplier $\hat{H}$ and lifts of a self-adjoint $\mathcal{A}$-multiplier H.*

Then the following K-theory classes are the same:

- *(i)* $[\mathcal{C}(\hat{H}_u)]_1^M - [\mathcal{C}(\hat{H}_v)]_1^M = [\mathcal{C}(\hat{H}_u)\mathcal{C}(\hat{H}_v)^*]_1 \in K_1(\mathcal{E})$.
- *(ii)* $[\mathbb{1} - e_+ + uv^*]_1 \in K_1(\mathcal{E})$ *where* $e_+ = uu^* = vv^*$.
- *(iii)* $[e^{\imath\pi(\chi_\Delta(\hat{H}_u)+1)}]_1 - [e^{\imath\pi(\chi_\Delta(\hat{H}_v)+1)}]_1 \in K_1(\mathcal{E})$ *for any switch function* χ_Δ *adapted to a bulk gap* Δ *of* H.

Proof. The classes from (i) and (ii) are well-defined since $u - v \in \mathcal{E}$ (where we as usual include $\mathcal{C}(\hat{H}_u) \in M(\hat{\mathcal{A}}) \otimes M_N(\mathbb{C}) \subset M(\hat{\mathcal{A}}) \otimes \mathbb{K}$). They are equal simply by expanding $\mathcal{C}(\hat{H}_v)^* = \mathcal{C}(\hat{H}_u)^* - u^* + v^*$. In the multiplier picture we have by definition

$$\begin{aligned}[\mathcal{C}(\hat{H}_u)]_1^M - [\mathcal{C}(\hat{H}_v)]_1^M &= [\mathcal{C}(\hat{H}_u)\mathcal{C}(\hat{H}_v)^*]_1^M - [\mathbb{1}]_1^M \\ &= [(\mathcal{C}(\hat{H}_u), \mathcal{C}(\hat{H}_v))]_1 \ \in K_1(\mathbb{P}(M^s(\mathcal{E}), \mathcal{E} \otimes \mathbb{K}) \simeq K_1(\mathcal{E}),\end{aligned}$$

but then by homotopy in $K_1(\mathbb{P}(M^s(\mathcal{E}), \mathcal{E} \otimes \mathbb{K}))$ one has

$$[\mathcal{C}(\hat{H}_u), \mathcal{C}(\hat{H}_v)]_1 = [f(\hat{H}_u), [f(\hat{H}_v)]_1$$

for any unitary function $f \in 1 + C_0(\mathbb{R})$ with the same winding number as $t \in \mathbb{R} \mapsto (t - \imath)(t + \imath)$, hence in particular

$$[\mathcal{C}(\hat{H}_u)]_1^M - [\mathcal{C}(\hat{H}_v)^*]_1^M = [e^{\imath\pi(\chi_\Delta(\hat{H}_u)+1)}]_1^M - [e^{\imath\pi(\chi_\Delta(\hat{H}_v)+1)}]_1^M$$

and we can drop the M on the r.h.s. since the elements are actually in $K_1(\mathcal{E})$ if there is a bulk gap. □

The most important statement in this Proposition is the equivalence between the classes in (ii) and (iii), since the former is related directly to the boundary conditions and the latter to the boundary states. In the chirally symmetric case the obvious analogue to compare two boundary conditions that respect the chiral symmetry is the class $[(\hat{e}_\Delta)_u]_0^M - [(\hat{e}_\Delta)_v]_0^M \in K_0(\mathcal{E})$ as in Proposition 5.2.1 which can be related to the difference $[\frac{1}{2}(Ju - \mathbb{1})]_0^M - [\frac{1}{2}(Jv - \mathbb{1})]_0^M \in K_0(\mathcal{E})$ which is then a difference of projections.
An important take-away here is that the boundary condition can encode a non-trivial topological content which forces the boundary invariants to differ for some boundary conditions that are not homotopic to each other, irrespective of any bulk topological invariants. Hence the simple bulk-boundary correspondence as one has it for completely affiliated Hamiltonians cannot generally be expected to hold for all self-adjoint boundary conditions. One should also note that the class of resolvent-affiliated extensions is only a small and well-behaved subset of all possible boundary conditions, in general one can say even less about the possibility of bulk-boundary correspondence.

A practical concern is that one usually wants to make the extension problem as simple as possible since the deficiency subspaces very quickly become complicated to parametrize. By Kato-Rellich dropping bounded operators does not affect the domain and it also does not affect the relative $K_1(\mathcal{E})$-class (even though the partial isometries describing the extension will change):

Lemma 5.2.6 *Let $\hat{H}_u$ be a self-adjoint extension of a symmetric $\hat{\mathcal{A}}$-multiplier $\hat{H}$ which is the lift of a resolvent-affiliated self-adjoint $\mathcal{A}$-multiplier H. For $\hat{V} = \hat{V}^* \in M(\hat{\mathcal{A}})$ the self-adjoint extensions $(\hat{H}+\hat{V})_{u'}$ of $\hat{H}+\hat{V}$ are in one-to-one correspondence with those of $\hat{H}$ in the sense that*

$$(\hat{H}+\hat{V})_{u'} = \hat{H}_u + \hat{V}$$

for a unique pair of partial isometries u', u. Moreover $(\hat{H}+\hat{V})_{u'}$ is resolvent-affiliated to $\hat{\mathcal{A}}$ if only if $\hat{H}_u$ is resolvent-affiliated. Given two pairs of resolvent-affiliated extensions related by

$$(\hat{H}+\hat{V})_{u'} = \hat{H}_u + \hat{V}$$
$$(\hat{H}+\hat{V})_{v'} = \hat{H}_v + \hat{V}$$

with $u - v \in \mathcal{E}$ then also $u' - v' \in \mathcal{E}$ and

$$[\mathbb{1} - e'_+ + u'(v')^*]_1 = [\mathbb{1} - e_+ + uv^*]_1.$$

Proof. The existence of u' for any u follows from Kato-Rellich and since the situation is symmetric under exchange $\hat{H}+\hat{V} \leftrightarrow \hat{H}$ there is a one-to-one correspondence. The statements about resolvent-affiliation follow from the resolvent identity

$$\frac{1}{(\hat{H}+\hat{V})_{u'}+\imath} = \frac{1}{(\hat{H})_u+\imath} + \frac{1}{(\hat{H})_u+\imath}\hat{V}\frac{1}{(\hat{H}+\hat{V})_{u'}+\imath}$$

which consists of terms in $\hat{\mathcal{A}}$ if $(\hat{H})_u$ is resolvent-affiliated. The resolvent-affiliation of $(\hat{H}+\hat{V})_{u'}$ and $(\hat{H}+\hat{V})_{v'}$ implies

$$\frac{\imath}{2}(u'-v') = \frac{1}{(\hat{H}+\hat{V})_{u'}+\imath} - \frac{1}{(\hat{H}+\hat{V})_{v'}+\imath} \in \mathcal{E}.$$

since both terms on the right-hand side are lifts of $(H+V+\imath)$. For the final statement we use the equivalent formulation of Corollary 5.2.4 in terms of the Cayley-Transforms and note that

$$t \in [0,1] \mapsto \big(\mathcal{C}(\hat{H}_u + t\hat{V}), \mathcal{C}(\hat{H}_v + t\hat{V})\big)$$

is a norm-continuous path of unitaries in $\mathbb{P}(M^s(\mathcal{E}), \mathcal{E} \otimes \mathbb{K})$, hence

$$\begin{aligned}[\mathcal{C}(\hat{H}_u)]_1^M - [\mathcal{C}(\hat{H}_v)]_1^M &= [\mathcal{C}(\hat{H}_u + \hat{V})]_1^M - [\mathcal{C}(\hat{H}_v + \hat{V})]_1^M \\ &= [\mathcal{C}((\hat{H} + \hat{V})_{u'})]_1^M - [\mathcal{C}((\hat{H} + \hat{V})_u)]_1^M.\end{aligned}$$

□

In some cases one has a reference Hamiltonian that does not have boundary states at least for certain boundary conditions, i.e. the lift $\hat{H}_0$ is sometimes also gapped. Then the boundary unitary corresponding to $\hat{H}_0$ vanishes in Corollary 5.2.2. We can now also say what happens for different boundary conditions:

Corollary 5.2.7 *Let H be a self-adjoint $\mathcal{A}$-multiplier with spectral gap Δ and reference Hamiltonian H_0 with $V = H - H_0 \in M(\mathcal{A})$ a bounded multiplier. Assume that there exists a self-adjoint lift $(\hat{H}_0)_u$ of H_0 resolvent-affiliated to $\hat{\mathcal{A}}$ and such that $(\hat{H}_0)_u$ does have a spectral gap in Δ. Let then $\hat{H}_{u'} = (\hat{H}_0)_u + \hat{V}$ be a lift of H with $\hat{V} \in M(\hat{\mathcal{A}})$ a self-adjoint lift of V corresponding to a self-adjoint extension of $\hat{H} = \hat{H}_0 + \hat{V}$.*

If $\hat{H}_{v'} = (\hat{H}_0)_v + \hat{V}$ is another self-adjoint lift of H such that $(\hat{H}_0)_v$ and $(\hat{H}_0)_u$ are self-adjoint extensions of a common symmetric $\hat{\mathcal{A}}$-multiplier $\hat{H}$ with

$$((\hat{H}_0)_v + \imath)^{-1} - ((\hat{H}_0)_u + \imath)^{-1} = \frac{\imath}{2}(u - v) \in \mathcal{E}$$

for two unitaries $u, v \in M(\mathcal{E})$ then

$$[e^{\imath\pi(\chi_\Delta(\hat{H}_{v'})+1)}]_1 = \operatorname{Exp}\left([\chi(H < \Delta)]_0^M - [\chi(H_0 < \Delta)]_0^M\right) - [uv^*]_1.$$

Proof. The relationship between the Hamiltonians is as follows

$$\begin{array}{ccc} (\hat{H}_0)_u & \longleftrightarrow & \hat{H}_{u'} \\ \updownarrow & & \updownarrow \\ (\hat{H}_0)_v & \longleftrightarrow & \hat{H}_{v'} \end{array}.$$

The boundary classes of top and bottom rows can be compared using Corollary 5.2.4 and Lemma 5.2.6 and of the left and right column using the relative bulk-boundary correspondence for resolvent-affiliated Hamiltonians Corollary 5.2.2.

The former gives

$$[e^{\imath\pi(\chi_\Delta((\hat{H}_0)_u)+1)}]_1 - [e^{\imath\pi(\chi_\Delta((\hat{H}_0)_v)+1)}]_1 = [uv^*]_1$$

and the latter

$$[e^{\imath\pi(\chi_\Delta(\hat{H}_{v'})+1)}]_1 - [e^{\imath\pi(\chi_\Delta((\hat{H}_0)_v)+1)}]_1 = \mathrm{Exp}\left([\chi(H<\Delta)]_0^M - [\chi(H_0<\Delta)]_0^M\right).$$

Inserting the assumption $[e^{\imath\pi(\chi_\Delta((\hat{H}_0)_u)+1)}]_1 = 0$ to solve for $[e^{\imath\pi(\chi_\Delta((\hat{H}_0)_v)+1)}]_1$ yields the result. □

In words, assume that we can restrict a reference Hamiltonian H_0 to a resolvent-affiliated halfspace Hamiltonian $(\hat{H}_0)_u$ with a boundary condition u for which there happens to be no spectrum in the bulk gap Δ. If we restrict a related bulk Hamiltonian H with that same boundary condition u then the class of the boundary unitary $[\hat{u}_\Delta]_1$ is exactly an image of the bulk-difference class of Fermi projections under the exponential map. If one instead of u uses a different boundary condition v for which resolvent-affiliation still holds, then the class of the boundary unitary is corrected by the class $[uv^*]_1$ which compares the two boundary conditions.

5.3 Examples

5.3.1 Tight-binding models

Here we discuss bulk-edge correspondence for the disordered non-commutative torus as a representative for all tight-binding models. For the bulk-boundary correspondence we follow [111] and consider systems with straight boundaries aligned to some hypersurface $v \cdot \mathbb{Z}^d > 0$ for some unit vector $v \in S^{d-1}$. This vector generates a linear flow on the torus via

$$\xi_t = \theta_{vt},$$

i.e. restriction of the gauge-action θ to a one-parameter subgroup. There are basically two different cases, one is that the components of v are rationally dependent in which case the action is periodic with some period Λ_v; hence ξ is a $\Lambda_v\mathbb{T}$-action. In the other case when v is not rationally dependent the range of $v\mathbb{R}$ is dense in $\mathbb{R}^d$.

In both cases a good exact sequence for bulk-boundary correspondence is the smooth Toeplitz extension

$$0 \to C(\mathbb{T}^d_{\mathbf{B},\Omega}) \rtimes_\xi G_v \to \mathrm{T}_+(C(\mathbb{T}^d_{\mathbf{B},\Omega}), \xi, G_v) \to C(\mathbb{T}^d_{\mathbf{B},\Omega}) \to 0 \tag{5.3.1}$$

with $G_v = \mathbb{R}$ in the rationally independent case and $G_v = \Lambda_v\mathbb{T}$ otherwise. With this choice (and under certain additional conditions on the ergodicity of the disorder space) one can show that $\mathrm{T}_+(\mathbb{T}^d_{\mathbf{B},\Omega}, \xi, G_v)$ is represented faithfully on the Hilbert space $\ell^2(\mathbb{Z}^d)$ in such a way that the functions of the abstract generator X correspond to functions of the position operator $X \cdot v$ on the lattice. The dual trace $\hat{\mathcal{T}}_\xi$ is naturally a trace per unit surface area [111, Section 5.2].

The crossed product with the continuous group G_v can make those algebras appear more complicated than they actually are, for the exact sequence (5.3.1) is isomorphic to the sequence

$$0 \to C_0(G_v, C(\Omega)) \rtimes_{T^v,\rho_\mathbf{B}} \mathbb{Z}^d \to C_{0,*}(G_v, C(\Omega)) \rtimes_{T^v,\rho_\mathbf{B}} \mathbb{Z}^d \to C(\Omega) \rtimes_{T^v,\rho_\mathbf{B}} \mathbb{Z}^d \to 0$$

with the group action (see the proof of [111, Proposition 5.2.1])

$$(T^v_x f)(r,\omega) = f(r - x \cdot v, T_x\omega), \qquad r \in G_v.$$

Hence the exact sequence simply arises from the $\mathbb{Z}^d$-equivariant exact sequence

$$0 \to C_0(G_v, C(\Omega)) \to C_{0,*}(G_v, C(\Omega)) \overset{\mathrm{ev}_\infty}{\to} C(\Omega) \to 0$$

with evaluation at $+\infty$. The bulk-boundary correspondence can then be expressed in terms of weak Chern number [103, 111]:

Theorem 5.3.1 *Let $h = h^* \in M_N(\mathbb{T}^d_{\mathbf{B},\Omega})$ have a spectral gap in Δ and let $\hat{h} = \mathcal{P}_+ h \mathcal{P}_+ + v \in M_N(T(C(\mathbb{T}^d_{\mathbf{B},\Omega}), \xi, G_v)$ be a self-adjoint lift with $v \in C(\mathbb{T}^d_{\mathbf{B},\Omega}) \rtimes_\xi G_v$ and $\mathcal{P}_+$ an approximate half-space projection as in Section 3.1. For $\hat{u}_\Delta$ as in Proposition 5.1.2 one has*

$$\langle Ch_{\mathcal{T},\alpha\times\xi}, [e_F]_0\rangle \;=\; \langle Ch_{\hat{\mathcal{T}}_\xi,\alpha}, [\hat{u}_\Delta]_1\rangle$$

for α any restriction of θ to an $\mathbb{R}^{2m-1}$-action with generators orthogonal to v.

If h and $\hat{h}$ satisfy are chirally symmetric then

$$\langle Ch_{\mathcal{T},\alpha\times\xi}, [u_F]_1\rangle \;=\; -\langle Ch_{\hat{\mathcal{T}}_\xi,\alpha}, [\hat{e}_\Delta]_0\rangle$$

for α any restriction of θ to an $\mathbb{R}^{2m}$-action with generators orthogonal to v.

Proof. An immediate consequence of Theorem 3.3.2 and Proposition 5.1.2 since a Hamiltonian of the form $\hat{h}$ is strongly affiliated. □

In the rationally independent case we can also construct a lift using a smooth switch function instead of the projection to a halfspace. However, since the boundary indices are protected by an index theorem, the numerical equality of Chern numbers also holds for sharp boundaries (though, one needs to be slightly careful about the algebras involved and in the end also their Hilbert space representations, we refer to [111, Section 5.4] for details). A nice feature of this bulk-boundary correspondence is that the dependence on the cutting angle is manifestly continuous even though the algebras and representatives vary.

One is here always in the situation of Proposition 5.1.2, since strong affiliation is the generic case for a unital bulk algebra. To get an example for a tight-binding model with only a relative bulk-boundary correspondence we merely have to revisit our substrate models from Section 4.3.5. Assume that we are in that situation where $H = H_0 + V$ with a $(d+1)$-dimensional substrate Hamiltonian H_0 and a d-dimensional surface insulator V that have a common spectral gap. If we now cut an additional surface into our system there are several distinct possibilities. In the easiest case the quarterspace Hamiltonian $\hat{H}_0$ still has a spectral gap in Δ. Then we expect the usual bulk-boundary correspondence, i.e. the quarterspace Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$ will have corner modes corresponding to the topological invariants of the surface Hamiltonian. On the other hand, it may happen that $\hat{H}_0$ already has topological surface modes, e.g. if H_0 was actually a weak topological insulator that only happened to have a gapped surface for some specific choice of halfspace cut. Then one must compare the surface states of $\hat{H}_0$ and $\hat{H}$ and can only relate e.g. the difference $[e^{\imath\pi(\chi_\Delta(\hat{H})+1)}]_1^M - [e^{\imath\pi(\chi_\Delta(\hat{H}_0)+1)}]_1^M$ with the bulk topological invariants (which where also relative and compared the Fermi projections of H and H_0). Loosely speaking, if one divides out the boundary modes already present for $\hat{H}_0$ alone, then what remains should be a surface-corner correspondence. The commutative diagram underlying the bulk-boundary correspondence is (1.5.2), or in this case

$$
\begin{array}{ccccc}
\mathcal{E}_d & \longrightarrow & \hat{\mathcal{A}}_d & \xrightarrow{\;q\;} & \mathcal{A}_d \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{P}(\mathrm{Ker}(\overline{q}), \mathcal{E}_d \otimes \mathbb{K}) & \to & \mathbb{P}(M^s(\hat{\mathcal{A}}_d), \hat{\mathcal{A}}_d \otimes \mathbb{K}) & \xrightarrow{\overline{q}\oplus\overline{q}} & \mathbb{P}(M^s(\mathcal{A}_d), \mathcal{A}_d \otimes \mathbb{K})
\end{array}
\tag{5.3.2}
$$

where $\mathcal{A}_d = C(\mathbb{T}^d_{\mathbf{B},\Omega})$ and $\mathcal{E}_d$, $\hat{\mathcal{A}}_d$ are boundary and half-space algebra for a d-dimensional system (such as a smooth Toeplitz extension), i.e. the algebras describing the topmost surface layer of the substrate. For the relative bulk-boundary correspondence one must always make sure that one only compares compatible Hamiltonians in $M^s(\hat{\mathcal{A}}_d)$, i.e. those whose difference lies in $\hat{\mathcal{A}}_d \otimes \mathbb{K}$ which encodes the condition that they must also describe almost identical substrates that are cut in almost the same way. Without that condition it is easy to generate apparent counterexamples to the conclusion of Proposition 5.2.1. Of course, taking two completely different substrates is out of the question since it yields plainly incomparable surface and hinge states, but there are also other subleties. Consider, for example, a quarter-space substrate in three dimensions which has a top surface described by $C(\mathbb{T}^2_{\mathbf{B},\Omega}) \otimes \mathbb{K}$ and another surface orthogonal to that, which we call the cut surface. Everything else being kept equal, modifying only the cut surface by covering it with different two-dimensional topological insulators yields Hamiltonians that have inequivalent hinge modes, even though the "bulk" (i.e. the top surface) stays the same.

The main takeaway is that deviations from bulk-boundary correspondence should not be considered exclusive to unbounded Hamiltonians since very similar phenomena appear also for bounded multipliers.

Let us also emphasize that the above exclusively concerns the case of a spectrally gapped bulk Hamiltonian. The case of a mobility or pseudogap requires very different methods.

5.3.2 Continuum models

For the bulk algebra $\mathcal{A} = C(\mathbb{R}^d_{\mathbf{B},\Omega})$ of continuum models we can also use a smooth Toeplitz extension for a one-parameter subgroup ξ of the dual action θ, generated by a normal vector v. Since the bulk algebra was already a crossed product one actually recovers here a Wiener-Hopf extension, since the Toeplitz extension is termwise isomorphic to

$$0 \to C_0(\mathbb{R}, C(\Omega)) \rtimes_{T^v,\rho_{\mathbf{B}}} \mathbb{R}^d \to C_{0,*}(\mathbb{R}, C(\Omega)) \rtimes_{T^v,\rho_{\mathbf{B}}} \mathbb{R}^d \to C(\Omega) \rtimes_{T,\rho_{\mathbf{B}}} \mathbb{R}^d \to 0.$$

To see this one compares with the discrete case above, the sequence here arises by taking termwise the twisted crossed of

$$0 \to C_0(\mathbb{R}, C(\Omega)) \to C_{0,*}(\mathbb{R}, C(\Omega)) \to C(\Omega) \to 0.$$

For halfspace problems the above exact sequence is less than ideal since the left and middle algebras are not supported only on the right halfspace but also leak into the left halfspace. We can eliminate this irrelevant part by defining new algebras $\hat{\mathcal{A}}, \mathcal{E}$ as the elements of the above halfspace respectively boundary algebra which are properly supported on the positive halfspace, i.e.

$$\hat{\mathcal{A}} = \{\hat{a} \in \mathrm{T}(\mathcal{A}, \mathbb{R}, \xi) : \hat{a} = \hat{a}P_+ = P_+\hat{a}\}, \quad \mathcal{E} = \hat{\mathcal{A}} \cap (\mathcal{A} \rtimes_\xi \mathbb{R})$$

for $P_+ = \chi(X_\xi > 0)$ the generator in the regular representation.

They form C^*-subalgebras and one has the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \hat{\mathcal{A}} & \longrightarrow & \mathcal{A} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{A} \rtimes_\xi \mathbb{R} & \longrightarrow & \mathrm{T}_+(\mathcal{A}, \mathbb{R}, \xi) & \longrightarrow & \mathcal{A} & \longrightarrow & 0 \end{array}$$

with exact rows, which tells us that we can still use the well-understood connecting maps of the smooth Toeplitz extension, at least as far as the pairings with Chern cocycles are concerned.

For a Schrödinger-type operator of the form Laplacian plus potential we are in the strongly affiliated situation:

Theorem 5.3.2 *Let $H = -\nabla^2 + V$ with $V = V^* \in M(C(\mathbb{R}^d_{\mathbf{B},\Omega}))$ have a spectral gap in Δ. H is strongly p-smooth for any $p \in (\frac{d}{2}, \infty]$ and resolvent-affiliated.*

Let $\widehat{\nabla^2}$ be a lift of ∇^2 to a self-adjoint $\hat{\mathcal{A}}$-multiplier that is resolvent-affiliated to $\hat{\mathcal{A}}$ and bounded from below. Then for any lift $\hat{V} \in M(\hat{\mathcal{A}})$ of V the self-adjoint multiplier

$$\hat{H} = -\widehat{\nabla^2} + \hat{V}$$

is a lift of H and strongly affiliated to $\hat{\mathcal{A}}$.

For $\hat{u}_\Delta$ as in Proposition 5.1.2 one has

$$\langle Ch_{\mathcal{T},\alpha\times\xi}, [e_F]_0\rangle \;=\; \langle Ch_{\hat{\mathcal{T}}_\xi,\alpha}, [\hat{u}_\Delta]_1\rangle$$

for α any restriction of θ to an $\mathbb{R}^{2m}$-action with generators orthogonal to v.

Proof. Since everything is bounded from below the strong affiliation follows immediately from the resolvent-affiliation. For the duality of Chern numbers one applies Theorem 3.3.2, the images under the boundary maps of the top sequence are also images under the the boundary maps of the bottom sequence. □

The remaining problem is to construct a resolvent-affiliated lift $\widehat{\nabla^2}$. One can guess that the Dirichlet-Laplacian is one such example and confirm that by computing the integral kernel of its resolvent. An alternative powerful method is to use norm-resolvent convergence. The Laplacian with domain wall $-\Delta + m\Theta$, Θ a smooth approximation of the projection to the negative halfspace is easily seen to be affiliated to the two-sided Toeplitz algebra. If one takes the limit of $m \to \infty$ and simultaneously of Θ to a sharp projection then the resolvents converge in a modified norm-resolvent sense and the result must therefore still be in the two-sided Toeplitz algebra. Since the limit vanishes on the negative halfpace, the resolvent of the limiting operator is moreover in $\hat{\mathcal{A}}$. Indeed, this procedure recovers exactly the resolvent of the Dirichlet-Laplacian on the positive halfspace. More analytic detail is relegated to the more technical Section 6.3.2 further below.

Another example which is strongly affiliated but not bounded from below is the two-dimensional regularized Dirac Hamiltonian (taken from [125])

$$H = \begin{pmatrix} m + \epsilon\nabla^2 & \imath\nabla_x + \nabla_y \\ \imath\nabla_x - \nabla_y & -m - \epsilon\nabla^2 \end{pmatrix}$$

which is strongly affiliated to the bulk algebra and also to the half-plane algebra $\hat{\mathcal{A}}$ when restricted with Dirichlet-Dirichlet boundary conditions

$$\psi_1(0) = 0 = \psi_2(0)$$

since the Dirichlet-Laplacian is strongly affiliated and the first order term can be shown to be irrelevant since it is infinitesimally bounded w.r.t. the Laplacian. The edge Chern number $\langle \mathrm{Ch}_{\hat{\mathcal{T}}_{e_2},e_1}, [\hat{u}_\Delta]_1 \rangle$ is $\frac{1}{2}(\mathrm{sgn}(m) + \mathrm{sgn}(\epsilon))$ equal to the bulk Chern number, as is consistent with Proposition 5.1.2. For any other self-adjoint boundary condition that also leads to a resolvent-affiliated multiplier we can conclude from Corollary 5.2.7 that for the central gap

$$\langle \mathrm{Ch}_{\mathcal{T},\theta}, [e_F]_0 \rangle \;=\; \langle \mathrm{Ch}_{\hat{\mathcal{T}}_\xi,\xi^\perp}, [\hat{u}_\Delta]_1 \rangle + \langle \mathrm{Ch}_{\hat{\mathcal{T}}_\xi,\xi^\perp}, [U_{\mathrm{BC}}]_1 \rangle$$

where U_{BC} is the unitary from Corollary 5.2.7 which compares Dirichlet and the new boundary condition. Notably the correction term from the boundary condition is independent of the sign of the mass by Lemma 5.2.6.

On the other hand, for the boundary condition [125]

$$\psi_1|_{y=0} = 0, \qquad -\imath\nabla_y\psi_2|_{y=0} = \nabla_x\psi_2|_{y=0}$$

one finds no edge states for neither sign of the mass term. Since the difference of bulk Chern numbers is 1 between positive and negative mass, that is a contradiction to relative bulk-boundary correspondence, hence we must conclude that the Hamiltonians are not resolvent-affiliated with those boundary conditions.

This is the first indication that far from every boundary condition that can be written down as a simple expression leads to resolvent-affiliated extensions. In the end, all depends on the structure of the deficiency subspaces which can in practice be very complicated to parametrize efficiently, if they can be computed analytically at all.

To discuss this in more detail we now specialize the extension theory lined out in Proposition 5.2.5 to Hamiltonians given by differential operators on $\mathbb{R}^d$. The bulk Hamiltonian H is a self-adjoint matrix differential operator resolvent-affiliated to $C(\mathbb{R}^d_{0,*}) \simeq C_0(\mathbb{R}^d)$ (and one can ignore a possible bounded potential here since it would not affect the extension problem). We make the Ansatz that the half-space Hamiltonian is translation-invariant w.r.t. the directions orthogonal to the boundary, thus via Fourier transform

$$\hat{H} = \int_{\mathbb{R}^{d-1}} \hat{H}_k \mathrm{d}k$$

where each $\hat{H}_k$ is a symmetric one-dimensional differential operator with domain $C_c^\infty(\mathbb{R}_+) \otimes \mathbb{C}^N$. Checking that such a symmetric operator defines a multiplier of the halfspace-algebra is not difficult in most cases: The problem can be solved fiberwise with some minimal input from the theory of differential operators and from then it is enough to require a Riesz-continuous dependence on the momentum k. The C^*-algebraic extension theory of Theorem 5.2.3 generically does apply for the cases considered in this work. Assume that the $\hat{H}_k$ have finite and equal deficiency indices

$$N_\pm = \dim \mathrm{Ker}(\hat{H}_k^* \mp \imath)$$

which do not depend on k. Then one can solve the extension problem for each fiber $\hat{H}_k$ and take the direct integral to obtain a self-adjoint extension of $\hat{H}$. The self-adjoint extensions $\hat{H}_u$ which are $\hat{\mathcal{A}}$-multipliers should then be in one-to-one correspondence with multipliers $u \in C_b(\mathbb{R}^{d-1}, \mathbb{K})$ where $u(k)$ is a partial isometry with initial and final projection given by the deficiency subspaces $\mathrm{Ker}(\hat{H}_k^* \pm \imath)$. By continuously identifying the deficiency subspaces with any N_+-dimensional vector space one can also view those as matrix functions $u \in C_b(\mathbb{R}^{d-1}, U_{N_+}(\mathbb{C}))$.

Now assume that there are two such u, v for which $\hat{H}_u$,$\hat{H}_v$ are resolvent-affiliated to $\hat{\mathcal{A}}$. Then we must have $u - v \in C_0(\mathbb{R}^{d-1}, M_N(\mathbb{C}))$ and the K-theory class $[\mathbb{1} - e_+ + uv^*]_1$ is equal to that of $uv^* \in \mathbb{1} + C_0(\mathbb{R}^{d-1}, U_N(\mathbb{C}))$ under some identification $[uv^*] \in K_1(C_0(\mathbb{R}^{d-1})) \simeq K_1(\mathcal{E})$. In the case $d = 2$ and $N > 0$, the unitary uv^* can be considered to be a loop with fixed point $\mathbb{1}$. One can always construct pairs u,v for which uv^* has an arbitrary winding number, hence leading to boundary conditions whose edge Chern numbers in a fixed bulk gap Δ differ by that same arbitrary integer. This means that we can practically always find topologically charged boundary conditions that lead to edge modes. Note, however, that due to Proposition 5.2.5 a non-trivial class $[uv^*]_1$ can only occur if the two extensions u, v lead to two Hamiltonians which have no common gap in their spectra. This possibility is therefore excluded automatically if we constrain the halfspace Hamiltonians to be bounded from below. Also large topological charges generally lead to very artificial boundary conditions that are unlikely to be relevant for physically interesting situations.

As a computable example we perform this analysis for the two-dimensional massive Dirac Hamiltonian (4.3.8). Restricting it to a halfspace one obtains a symmetric multiplier. It is enough to compute the deficiency subspaces for the massless case $m = 0$ due to Lemma 5.2.6, in particular the mass term does not affect the eventual Hilbert space domain of the extensions by Kato-Rellich. A possible parametrization after partial Fourier transform w.r.t. the orthogonal direction is

$$\hat{H}_{k_x} = \begin{pmatrix} 0 & k_x + \nabla_y \\ k_x - \nabla_y & 0 \end{pmatrix}$$

and one has the one-dimensional deficiency subspaces $\mathrm{Ker}(\hat{H}_{k_x}^* \mp \imath)$ spanned by the vectors

$$\psi_\pm(k_x, y) = \begin{pmatrix} \pm\imath(\sqrt{k_x^2 + 1} - k_x) \\ 1 \end{pmatrix} e^{-\sqrt{k_x^2+1}y}, \qquad \hat{H}_{k_x}\psi_\pm = \pm\imath\psi_\pm.$$

Any unitary function $\gamma : \mathbb{R} \to S^1$ specifies a self-adjoint extension by imposing that any $\varphi = (\varphi_1, \varphi_2) \in L^2(\mathbb{R} \times \mathbb{R}_+, \mathbb{C}^2)$ in the domain of the extension $\hat{H}_u$ satisfy

$$\varphi(k_x, 0) = \beta\psi_+(k_x, 0) + \beta\gamma(k_x)\psi_-(k_x, 0)$$

for some $\beta \in \mathbb{C}$ which leads to the boundary condition $\varphi_1(k_x, 0) = \alpha(k_x)\varphi_2(k_x, 0)$ with

$$\alpha(k_x) := \frac{\imath}{\sqrt{k_x^2 + 1} - k_x} \frac{\gamma + 1}{\gamma - 1}.$$

For simplicity we at first assume that α is a constant independent of k_x, then the extension is obtained by the unitary function

$$\gamma_\alpha(k_x) = \frac{\imath + f(k_x)\alpha}{-\imath + f(k_x)\alpha}$$

with the non-negative function $t(k_x) = \sqrt{k_x^2 + 1} - k_x$. Notably α must be real and $f(k_x)\alpha$ is up to a factor the inverse Cayley transform of the unitary γ_α. We find that $\gamma_\alpha(k_x)$ tends to 1 for $k_x \to -\infty$ and to -1 for $k_x \to \infty$ for all $\alpha \notin \{0, \infty\}$.

Similar to the Dirichlet boundary conditions for the Laplacian one can also single out resolvent-affiliated boundary conditions for the Dirac-Hamiltonian as infinite-mass boundary conditions. They are obtained by taking the norm-resolvent limit of the domain-wall Dirac-Hamiltonians [14]

$$H + \mu\sigma_3\chi$$

for $\mu \to \pm\infty$ with χ the indicator function of the negative halfspace. Using the methods of Section 6.3.2 one can show that χ can be replaced by a family of smooth switch functions, thus the limit is resolvent-affiliated to $\hat{\mathcal{A}}$. The infinite-mass boundary conditions coincide with the case $\alpha = \pm 1$ given above. From the discussion above, respectively Proposition 5.2.5, we know that this fixes the asymptotic behavior of $\gamma(k)$ for all resolvent-affiliated extension and conclude that the resolvent-affiliated extensions are in one-to-one correspondence with unitary functions that tend to ∓ 1 for $k_x \to \pm\infty$. For the special cases $\alpha = \infty$ and $\alpha = 0$ one also obtains self-adjoint extensions, but they are not resolvent-affiliated according to this characterization.

We obtain a simple consequence from our theory

Corollary 5.3.3 *Consider the massive Dirac Hamiltonian*

$$\hat{H}_m = \begin{pmatrix} m & \imath\nabla_x + \nabla_y \\ \imath\nabla_x - \nabla_y & -m \end{pmatrix}$$

restricted to the positive half-plane with the self-adjoint boundary condition

$$\psi_1|_{y=0} = \alpha\psi_2|_{y=0},$$

for some $\alpha \in \mathbb{R} \setminus \{0\}$. For $m \neq 0$ one of $\hat{H}_m$ or $\hat{H}_{-m}$ must have a non-vanishing edge Chern number.

Proof. The relative Chern number in the bulk is

$$\langle \mathrm{Ch}_{\mathcal{T},e_1\times e_2}, [p_m]_0^M - [p_{-m}]_0^M \rangle = 1$$

hence the same is true for the difference of edge Chern numbers

$$\langle \mathrm{Ch}_{\hat{\mathcal{T}}_{e_2},e_1}, [(\hat{u}_\Delta)_m]_1 - [(\hat{u}_\Delta)_{-m}]_1 \rangle = 1.$$

□

If one actually computes the edge states one finds that precisely one of $\hat{H}_m$ or $\hat{H}_{-m}$ has an edge mode and the other does not [62]. Also the edge mode jumps from one branch to the other precisely at the critical value $\alpha = 0$ where resolvent-affiliation is violated (which is also the only possible value due to the otherwise norm-continuous dependence of the resolvent on α). With this theory we cannot predict which sign of α produces an edge mode for which sign of m, however, what one can prove is that the association flips when α changes sign.

To see this we determine the unitary $[uv^*]_1$ from Proposition 5.2.5 comparing two boundary conditions. For $\alpha < 0$ the function $k_x \mapsto \gamma_\alpha(k_x) \in S^1$ describes a semicircle in the lower complex half-plane and for $\alpha > 0$ a semicircle in the upper half-plane. Thus the unitary $[uv^*]_1 \simeq [\gamma_\alpha \gamma_{\tilde{\alpha}}^*]_1$ has winding number ± 1 if and only if α and $\tilde{\alpha}$ have opposite sign. This winding number encodes exactly the difference of edge Chern numbers between different self-adjoint extensions, hence we know that if there is no edge mode for $\alpha < 0$ then one must appear for $\alpha > 0$ and vice versa.

We can even be more extreme and choose an artificial boundary condition where the function γ winds $\frac{2N+1}{2}$ times around 0. For such a boundary condition one must obtain edge Chern numbers N and $N+1$ for different signs of the mass term. In particular we can produce an arbitrarily large number of edge modes.

The resolvent convergence for the Laplacian and Dirac operators with domain walls was crucial in the discussion above to determine the resolvent-affiliated halfspace operators. Since this is possibly just a convenient coincidence let us also sketch a more general procedure which can be used to find resolvent-affiliated half-space Hamiltonians and also to independently verify the resolvent-affiliation. Let H be a self-adjoint resolvent-affiliated $\mathcal{A}$-multiplier and let $\hat{H}$ be the restriction of H to the domain $C_c(\mathbb{R}^{d-1}\times(\mathbb{R}\setminus\{0\}))$. Then $\hat{H}$ is a symmetric $\mathrm{T}(\mathcal{A},\mathbb{R},\xi)$-multiplier for ξ the dual (momentum-space) translation in the d-direction. Clearly, $\hat{H}$ has a self-adjoint extension that is resolvent-affiliated to $\mathrm{T}(\mathcal{A},\mathbb{R},\xi)$, namely H itself. That extension corresponds to a partial isometry $u\in M(\mathrm{T}(\mathcal{A},\mathbb{R},\xi))$ with $H=\hat{H}_u$. Now the idea is to construct a perturbation v of u such that $v-u\in\mathrm{T}(\mathcal{A},\mathbb{R},\xi)$ and the two halfspaces are decoupled in the sense that $\hat{H}_v$ commutes with the halfspace projections $P_\pm$. Then $P_+\hat{H}_v$ is resolvent-affiliated to $\hat{\mathcal{A}}$ by construction (and therefore determines all such $\hat{\mathcal{A}}$-resolvent-affiliated extensions up to perturbation).

We can give more details in the translation-invariant case

$$\hat{H}_0=\int_{\mathbb{R}^{d-1}}\hat{H}_k\mathrm{d}k$$

with one-dimensional symmetric differential operators $\hat{H}_k$. Since each $\hat{H}_k$ acts on two disjoint half-lines one can decompose the kernels

$$\mathrm{Ker}(\hat{H}_k^*\pm\imath)=:K_\pm=P_+K_\pm\oplus P_-K_\pm$$

with the subspaces supported on the range of the halfspace projections $P_\pm$. Often $N:=\dim(P_\mp K_\pm)=\dim(P_\pm K_\pm)<\infty$ with N independent of k. The self-adjoint extensions correspond to functions $k\in\mathbb{R}^{d-1}\mapsto u(k)$ valued in the partial isometries $K_+\to K_-$ and clearly the extension commutes with $P_\pm$ if and only if u maps $P_\pm K_+$ into $P_\pm K_-$, i.e. if u is interpreted as a matrix in $M_{2N}(\mathbb{C})\otimes C_b(\mathbb{R}^{d-1})$ then it should be block-diagonal. For example, if $H=-\nabla^2$ is the d-dimensional Laplacian then

$$\mathrm{Ker}(\hat{H}_k^*+\imath)=\mathrm{span}\{e^{-\sqrt{\imath+k_x^2}y}\chi(y>0)\}\oplus\mathrm{span}\{e^{\sqrt{\imath+k_x^2}y}\chi(y<0)\}$$

and

$$\mathrm{Ker}(\hat{H}_k^* - \imath) = \mathrm{span}\{e^{-\sqrt{-\imath+k_x^2}y}\chi(y>0)\} \oplus \mathrm{span}\{e^{\sqrt{-\imath+k_x^2}y}\chi(y<0)\}.$$

The transparent boundary condition $H = \hat{H}_u$ is obtained for a multiplier u which has a matrix representation $U \in -\mathbb{1}_2 + M_2(\mathbb{C}) \otimes C_0(\mathbb{R}^{d-1})$ w.r.t. to the deficiency subspaces. Dirichlet boundary conditions decouple the halfspaces and correspond to $v = -\mathbb{1}_2$, hence $u - v \in \mathcal{E}$ works out as expected.

Finally, let us also discuss the shallow-water Hamiltonian (4.3.10). Computing its spectrum for various boundary conditions one finds anomalies that are incompatible with some formulations of bulk-boundary correspondence [125], though obviously they should not be in contradiction to proven theorems. A complication in trying to apply the formalism of this chapter is that resolvent-affiliation cannot hold for the halfspace Hamiltonians since it does not hold in the bulk. Nevertheless, the bulk-boundary correspondence for strongly affiliated Hamiltonians may still be applicable (though difficult given that the bounded transform is hard to compute) and also the relative bulk-boundary, especially for extensions where the halfspace Hamiltonian is almost resolvent-affiliated in the sense that

$$(\hat{H} + \imath)^{-1} \in M_3(\hat{\mathcal{A}}^{\sim}). \tag{5.3.3}$$

Unfortunately, we cannot identify any strongly affiliated halfspace restrictions here. The obvious guess would be Dirichlet-Dirichlet boundary conditions, i.e. for $\phi \in W_2^1(\mathbb{R} \times \mathbb{R}_+) \oplus W_2^2(\mathbb{R} \times \mathbb{R}_+)$ one imposes

$$\phi_2|_{y=0} = 0, \qquad \phi_3|_{y=0} = 0, \tag{5.3.4}$$

and ϕ_1 is unrestricted since the Hamiltonian only acts on it through first-order terms. However, for this boundary condition the bulk-boundary relation of Proposition 5.1.2 is violated since there is for $f, \epsilon > 0$ only a single chiral gap-filling mode [125], hence the edge Chern number is 1 while the bulk Chern number is 2. Hence there is no strong affiliation. For $\hat{H}_{\mathrm{DD}}$ the self-adjoint halfspace restriction under (5.3.4) one has indeed the almost resolvent affiliation as in (5.3.3) since

$$\begin{aligned}(\hat{H}_{\mathrm{DD}} + \imath\mu)^{-1} &= (\hat{H}_0 + \imath\mu)^{-1} + (\hat{H}_0 + \imath\mu)^{-1}\hat{V}(\hat{H}_{\mathrm{DD}} + \imath\mu)^{-1} \\ &= \sum_{k=0}^{\infty} (\hat{H}_0 + \imath\mu)^{-1}(\hat{V}(\hat{H}_0 + \imath\mu)^{-1})^k\end{aligned}$$

with convergence in operator norm for $|\mu| > 1$ and the $\hat{\mathcal{A}}$-multipliers

$$\hat{H}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\imath(f - \epsilon\nabla^2_{\mathrm{DD}}) \\ 0 & \imath(f - \epsilon\nabla^2_{\mathrm{DD}}) & 0 \end{pmatrix}, \qquad \hat{V} = \begin{pmatrix} 0 & -\imath\nabla_x & -\imath\nabla_y \\ -\imath\nabla_x & 0 & 0 \\ -\imath\nabla_y & 0 & 0 \end{pmatrix}.$$

Thus $(\hat{H}_{\mathrm{DD}} + \imath)^{-1} \in M_3(\hat{\mathcal{A}}^{\sim})$ since each term lies in $M_3(\hat{\mathcal{A}}^{\sim})$. From the series expansion one can also read off that

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2\imath f \\ 0 & -2\imath f & 0 \end{pmatrix} (\hat{H}_{\mathrm{DD}} + \imath)^{-1} \in M_3(\hat{\mathcal{A}})$$

which implies that given two Dirichlet-Dirichlet Hamiltonians for which the Coriolis parameter f takes values of opposite signs one also has

$$F(\hat{H}_{\mathrm{DD}}^{(f<0)}) - F(\hat{H}_{\mathrm{DD}}^{(f>0)}) \in M_3(\hat{\mathcal{A}})$$

since the difference of bounded transforms lies in the same norm-closed algebra as the resolvent difference. We can hence apply the relative bulk-boundary correspondence Proposition 5.2.1 to conclude that for $f < 0$ one must have at least one chiral gap-filling mode whose chirality is opposite to that for $f > 0$, such that

$$\begin{aligned} &\langle \mathrm{Ch}_{\hat{\mathcal{T}}_{e_2}, e_1}, [\hat{u}_\Delta^{(f>0)}]_1 - [\hat{u}_\Delta^{(f<0)}]_1 \rangle \\ &\quad = \langle \mathrm{Ch}_{\mathcal{T}, e_1 \times e_2}, [\chi(H^{(f>0)} < 0)]_0 - [\chi(H^{(f<0)} < 0)]_0 \rangle = 2. \end{aligned}$$

Based on the computations of edges states in [125, 59] there are also many boundary conditions for which this almost resolvent affiliation cannot hold (this can be seen most easily when the Chern numbers of $\hat{u}_\Delta$ would have to vary within a fixed gap, i.e. depend on the choice of switch function). Without the resolvent condition it is even more difficult to verify the conditions of Proposition 5.2.1 since the bounded transform is not readily computable. Thus we cannot make any definitive statements for the vast majority of boundary conditions at this point. Let us again emphasize that there are no manifest contradictions to any proven statements of bulk-boundary correspondence, the problem at the heart of the matter is rather that for most boundary conditions it is difficult to predict if the edge topological invariants are well-defined at all, what the relation between bulk and edge topological invariants is and if it is possible to repair mismatches by taking additional topological quantities into account. The tools developed in this chapter can provide answers to those questions in more cases than before, but they require algebraic affiliation conditions that are difficult to check in practice (and sometimes simply do not apply).

5.4 Bulk-interface correspondence

In this section we discuss some of the particularities of bulk-interface correspondence, where one considers topological interface states between two systems. On the abstract level there is no difference between bulk-interface and bulk-boundary correspondence, the former arises from the latter simply by consideration of a bulk algebra $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$ which is the direct sum of two observable algebras. For that reason one also cannot say much more about the problem in general. Therefore we choose the concrete model of the two-sided Toeplitz extension of Chapter 3, namely

$$0 \to \mathcal{A} \rtimes_\xi G \hookrightarrow \mathrm{T}(\mathcal{A}, G, \xi) \overset{q}{\to} \mathcal{A}_- \oplus \mathcal{A}_+ \to 0 .$$

where $\mathcal{A}_\pm = \mathcal{A}$ is an observable algebra (different algebras can be treated, but require a different exact sequence, see for example [43]). For ξ a 1-parameter subgroup of the natural action θ one easily finds more concrete pictures of this exact sequence above in the case of $\mathcal{A} = C(\mathbb{T}^d_{\mathbf{B},\Omega})$ respectively $\mathcal{A} = C(\mathbb{R}^d_{\mathbf{B},\Omega})$ and can convince oneself that in representations on a physical Hilbert space it really describes an interface setup where two bulk observables in $\mathcal{A}$ are continuously interpolated.

The connecting maps of the sequence are as stated in Section 3.2 obtained as the difference between the connecting maps of two one-sided Toeplitz extensions; hence one has in particular the expected duality of Chern numbers if one interpolates two bulk Hamiltonians $H_\pm \in \mathcal{A}_\pm$ (as has also been found for two-dimensional interfaces in [78]).

The problem becomes more interesting if one allows multipliers as Hamiltonians, then we should use a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A} \rtimes_\xi G & \longrightarrow & \mathrm{T}(\mathcal{A},\xi,G) & \xrightarrow{q} & \mathcal{A}_- \oplus \mathcal{A}_+ & \longrightarrow & 0 \\ & & \downarrow\rho & & \downarrow\rho & & \downarrow\rho_+ + \rho_- & & \\ 0 & \to & \mathcal{A} \rtimes_\xi G \otimes \mathbb{K} & \to & \mathrm{T}^M(\mathcal{A}\otimes\mathbb{K},\xi,G) & \xrightarrow{q} & \mathbb{P}(M^s(\mathcal{A}),\mathcal{A}\otimes\mathbb{K}) & \to & 0 \end{array} \tag{5.4.1}$$

for the multiplier Toeplitz extension of Section 3.4. Using that extension we can extend the bulk-interface correspondence naturally to situations where the gapped bulk invariants arise in the multiplier picture by comparison with a reference Hamiltonian.

Proposition 5.4.1 *Let (H_-, H_+) be a pair of self-adjoint $\mathcal{A}$-multipliers with common spectral gap Δ and $F(H_+) - F(H_-) \in \mathcal{A}$. Let $\hat{H}_I$ be a self-adjoint $T(\mathcal{A},\xi,G)$-multiplier with $F(\hat{H}_I) \in T^M(\mathcal{A},\xi,G)$ and a lift of (H_-, H_+) under q. Then*

$$[e^{\iota(\chi_\Delta(\hat{H}_I)+1)}]_1 = \mathrm{Exp}_G^\xi\left([\chi(H_+ < \Delta)]_0^M - [\chi(H_+ < \Delta)]_0^M\right)$$

with Exp_G^ξ *defined as in Section 3.2 and similarly for the even K-theory class if $\hat{H}_I$ has a chiral symmetry.*

Proof. For any rank one projection $f \in \mathbb{K}$ the image of the bulk Fermi projections $[(\chi(H_+ < \Delta), \chi(H_- < \Delta)) \otimes f]_0 \in K_0(\mathbb{P}(M^s(\mathcal{A}), \mathcal{A}\otimes\mathbb{K}))$ under the bottom exact sequence of (5.4.1) is $[e^{\iota(\chi_\Delta(\hat{H}_I)+1)} \otimes f + \mathbb{1} - f]_1 \in K_1(\mathcal{A} \rtimes_\xi G \otimes \mathbb{K})$. By Proposition 3.4.2 and naturalness of the connecting maps we have

$$[e^{\iota(\chi_\Delta(\hat{H}_I)+1)}]_1 = \mathrm{Exp}_G^\xi(x)$$

for any $x \in K_0(\mathcal{A})$ such that $(\rho_+)_*(x) = [(\chi(H_+ < \Delta), \chi(H_- < \Delta))]_0$. By definition this is the element $[\chi(H_+ < \Delta)]_0^M - [\chi(H_+ < \Delta)]_0^M \in K_0(\mathcal{A})$. □

The setup one should think of is again that of two bulk Hamiltonians, i.e. self-adjoint $\mathcal{A}$-multipliers $H_\pm$, which differ by a smooth perturbation $V := H_+ - H_- \in M(\mathcal{A})$. An interface which interpolates between the two Hamiltonians in real space, can be e.g. chosen in the form

$$\hat{H}_I = H_- + \frac{1}{2}\{\mathcal{P}, V\} + \hat{e}$$

with $\mathcal{P}$ a smooth approximate projection to the positive half-space and $\hat{e}$ an interface term which vanishes at $\pm\infty$. They must be such that in the bounded picture

$$F(\hat{H}_I) = F(H_-) + \frac{1}{2}\{\mathcal{P}, F(H_+) - F(H_-)\} + \tilde{e} \in \mathrm{T}^M(\mathcal{A}, \xi, G) \qquad (5.4.2)$$

with $\tilde{e} \in \mathcal{E}$. That is in particular the case if $(H_\pm + \imath)^{-1} \in \mathcal{A}$ and $\hat{e} \in \mathcal{E}$. Hence the theory applies to interface Hamiltonians of the form $\hat{H}_I = H_0 + V$ with H_0 a differential operator and V a potential that interpolates between two asymptotic potentials $V_\pm$ at $\pm\infty$. Let us also emphasize that, as seen in Corollary 3.4.3, the Chern numbers are dual with respect to this exact sequence. For example, in a tight-binding model over $C(\mathbb{T}^d_{\mathbf{B},\Omega})$ connecting two spectrally gapped Hamiltonians with different Chern numbers will therefore usually result in boundary states (this can depend on the normal vector of the boundary, however, if one deals with weak Chern numbers). For a multiplier example, one can interpolate two two-dimensional Dirac Hamiltonians (4.3.8) of opposite mass across an interface to obtain a stable edge mode with edge Chern number ± 1 in accordance with the Chern number of the difference class $[p_m]_0^M - [p_{-m}]_0^M$ of the bulk Fermi projections. This latter fact has also been proven using very different methods in [13].

In the previous section we found that bulk-boundary correspondence can become complicated and unsatisfactory in the non-unital case due to the different boundary conditions and the necessity to consider relative topological invariants which limit our predictive power when considering a single system of interest. It is therefore nice to see that most difficulties disappear in the case of such smooth interfaces.

Nevertheless, they resurface if one also considers singular interfaces for which the bounded transform takes the form (5.4.2). By singular we mean that one starts with a symmetric multiplier $\hat{H}$ that is given by the restriction of two bulk-Hamiltonians $\hat{H} = H_+|_{\mathcal{D}_+} \oplus H_+|_{\mathcal{D}_-}$ to two halfspaces and looks at self-adjoint extensions $\hat{H}_I$ which lie in $\mathrm{T}^M(\mathcal{A}, \xi, G)$. That is, the potential jumps discontinuously at some hypersurface which is accommodated by choosing certain matching conditions for the domain. Similarly as we started above for the case of halfspaces one can again try to describe all resolvent-affiliated self-adjoint extensions for which bulk-interface correspondence works as expected. That is in general a difficult problem; in particular it is at least as difficult as the halfspace problem, since it includes as a special case all cases of decoupled halfspaces. One should note also that one can substantially lessen the continuity requirements at the boundary if one is only concerned about Chern numbers, since those are additionally protected by semifinite index theorems.

Let us finally note that a special case of the above setup arises for the trivial action ξ. Then $\mathrm{T}(\mathcal{A}, \mathbb{R}, \xi) = C_{*,*}(\mathbb{R}, \mathcal{A})$ consists of continuous paths of observables. A lift of a pair of Hamiltonians (H, H_0) consists then of a path $t \in \mathbb{R} \mapsto \hat{H}_t$ where $F(\hat{H}_t) - F(\hat{H}_0) \in \mathcal{A}$ converges at $\pm\infty$. Since the connecting maps are isomorphisms in this case, one finds that if H, H_0 have the common spectral gap Δ then the gap must close along the path if the difference class $[\chi(H < \Delta)]_0^M - [\chi(H_0 < \Delta)]_0^M$ does not vanish. If one likes, one may consider the class $[\chi(H < \Delta)]_0^M - [\chi(H_0 < \Delta)]_0^M$ as precisely the obstruction to gap opening along a straight-line path between $F(H)$ and $F(H_0)$, which can be seen as a $K_1(S\mathcal{A}) \simeq K_0(\mathcal{A})$-valued spectral flow. From that perspective relative bulk-boundary correspondence relates bulk spectral flows to differences in boundary invariants.

6 Nonsmooth bulk-boundary correspondence

In a mobility gap or a pseudogap the K-theoretic approach to bulk-boundary correspondence becomes difficult or even even impossible to apply. While one can in principle still write down exact sequences which are large enough to contain e.g. the Fermi projection/unitary of a mobility gapped Hamiltonian, the main problem is that one cannot find a good representative for a boundary class, since the absence of a bulk gap means that bulk and boundary states are mixed. Indeed, it is not even clear what the eventual fate of the boundary modes is; there is some hope that one may be able to prove the existence of absolutely continuous spectrum for the halfspace Hamiltonian and therefore long-range transport in the region of the bulk mobility gap [26] but no mathematical argument is in sight. It is also likely that the boundary states hybridize with the bulk states and are much less stable than the boundary states of a spectrally gapped insulator. Few rigorous results are available on the bulk-boundary correspondence of mobility gapped insulators [51, 109, 60, 119, 22]. A similar problem is the bulk-boundary correspondence for weak Chern numbers of pseudogapped semimetals. To some degree it can be treated for translation-invariant models by applying a partial Fourier transform to map a semimetal-Hamiltonian to a parametrized family of lower-dimensional Hamiltonians, almost all of which have a spectral gap. However, this picture is not stable for arbitrary (disordered) boundaries, and only works for very special bulk Hamiltonians. For disordered Hamiltonians the only results available appear to be those of [111] which are a special case of what we prove in the following. Let us also note that our results concern weak Chern numbers of semimetals and their corresponding edge states. There are also unrelated approaches which show that more subtle invariants of Dirac-type operators also lead to protected boundary modes [128, 36].

In this chapter we discuss the bulk-boundary correspondence first for one- and then for two-dimensional (weak) Chern numbers using two completely different approaches. For the one-dimensional case we use an index-theoretic approach which is based on an accidental coincidence of two algebras, namely the von Neumann algebra generated by the boundary observables and the classifying algebra for the one-dimensional Chern numbers from Chapter 2. It constitutes a generalization of one of the main results of [111] to a certain class of unbounded Hamiltonians. In fact, after the preparatory work of Chapter 4 the proofs and results could be adapted with only minor changes.

On the other hand, for the case of (weak) two-dimensional invariants we follow closely the approach of [51], which is to introduce a regularization of the chiral surface conductivity and show that it recovers the numerical bulk Chern numbers in a certain limit.

6.1 Flat bands of edge states

In this section we treat bulk-boundary correspondence for one-dimensional (weak) Chern numbers under limited smoothness conditions. Compared to the smooth algebraic bulk-boundary correspondence as one obtains it from a smooth Toeplitz extension as in the previous chapter one can prove a stronger statement, namely the edge states corresponding to a non-trivial one dimensional Chern number in the bulk are actual zero-energy eigenvalues (which are thus infinitely degenerate in higher dimension). Moreover, this does not require a spectral gap in the bulk, a mobility gap (and sometimes a pseudogap) are enough.

We begin with a tracial dynamical system $(\mathcal{A}, \theta, \mathcal{T})$ for bulk observables, though the Fermi projections and other related operators may only lie in the von Neumann algebra $L^\infty(\mathcal{A})$. The halfspace models are as in the examples of Chapter 5: A restriction ξ of θ to a one-parameter subgroup G_v, isomorphic to either $\mathbb{R}$ or $\mathbb{T}$ gives rise to the halfspace von Neumann algebra

$$\mathcal{N}_\xi := \mathbf{P}_\xi(L^\infty(\mathcal{A}) \rtimes_\xi G)\mathbf{P}_\xi$$

where $\mathbf{P} = \chi(X_\xi > r)$ is a positive spectral projection of the generator for an arbitrary but fixed value r that is not an eigenvalue of X_ξ (but it may be in the spectrum). One also has a dual trace $\hat{\mathcal{T}}_\xi$ which can be interpreted as a trace per unit surface area. In the following we suppress the dependence on ξ to write $\mathcal{N}$ and $\mathbf{P}$ for simplicity. We also make use of a partition of unity which decomposes the entire space into finite strips:

$$\mathbf{P} = \sum_{l\in\mathbb{N}} \chi(X_\xi \in [l, l+1)) =: \sum_{l\in\mathbb{N}} \mathbf{P}_l.$$

Our Hamiltonians in the bulk can be differential operators and hence they might not restrict easily to a halfspace; a choice of boundary condition may be necessary. To avoid difficult analytical questions in that regard we some (possibly unnecessarily strong) assumptions.

Assumption 6.1.1 *Let H be a strongly affiliated $\mathcal{A}$-multiplier which has a chiral symmetry, i.e. it anti-commutes with a self-adjoint unitary matrix J, $H = -JHJ$, and does not have 0 as an eigenvalue.*

We assume further that H is strongly p-smooth for all $p_0 \leq p \leq \infty$ with some $1 \leq p_0 \leq 2$ and satisfies the conditions of Theorem 4.3.23, i.e. in particular its Fermi projection lies in (a matrix unitization over) $W^1_p(\mathcal{A}, \xi)$ for some $1 < p \leq 2$ and $\mathbf{P}u_F\mathbf{P} + \mathbb{1} - \mathbf{P}$ is $\hat{\mathcal{T}}_\xi$-Fredholm with

$$\hat{\mathcal{T}}_\xi\text{−Ind}(\mathbf{P}u_F\mathbf{P} + \mathbb{1} - \mathbf{P}) = Ch_{\mathcal{T},\xi}(u_F^* - s(u_F)^*, u_F - s(u_F)) =: Ch_{\mathcal{T},\xi}(u_F).$$

The halfspace Hamiltonian $\hat{H}$ shall be a θ-smooth chirally symmetric self-adjoint operator affiliated to $\mathcal{N}$ and a lift of H in the sense that

$$F(\hat{H}) = \mathbf{P}F(H)\mathbf{P} + \hat{v}$$

with $\hat{v} \in \mathcal{K}(\mathcal{N})$ a boundary term such that

$$\|\mathbf{P}_m\hat{v}\|_{L^p(\mathcal{N})} \leq c_{k,p}\langle m\rangle^{-k}$$

for each $k \in \mathbb{N}$, $p \in (p_0, \infty]$.

Here we understand the θ-smoothness as being strictly smooth w.r.t. to the generators of θ in the sense of Definition 1.4.11 in some faithful Hilbert space representation of $\mathcal{N}$. It implies e.g. that the bounded transform $F(\hat{H})$ is smooth w.r.t. θ. Examples for bounded and unbounded Hamiltonians which satisfy those assumptions will be given in Section 6.3.

The strong affiliation is a rather restrictive assumption but it is needed since we can otherwise only have a relative bulk-boundary correspondence and the given regularity conditions are apparently not strong enough to compare the polar decompositions of two halfspace models with each other (unless they both have a bulk-gap, in which case the theory of Section 5 applies).

The main technical difficulty of this section is to prove that under certain conditions the splitting

$$\operatorname{sgn}(\hat{H}) = \mathbf{P}\operatorname{sgn}(H)\mathbf{P} \mod \mathcal{K}(\mathcal{N}) \tag{6.1.1}$$

holds. Let us first finish the proof of bulk-boundary correspondence under that assumption:

Proposition 6.1.2 *For $\hat{H}$ and H as in Assumption 6.1.1 if the splitting condition (6.1.1) holds then*

$$\hat{\mathcal{T}}_\xi(J\mathrm{Proj}_{\mathrm{Ker}(\hat{H})}) = Ch_{\mathcal{T},\xi}(u_F).$$

Proof. Due to the chiral symmetry one has

$$\mathrm{sgn}(\hat{H}) = \begin{pmatrix} 0 & \hat{U}^* \\ \hat{U} & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

with the off-diagonal component a partial isometry $\hat{U}$ such that

$$\hat{\mathcal{T}}_\xi(J\mathrm{Proj}_{\mathrm{Ker}(\hat{H})}) = \hat{\mathcal{T}}_\xi(\mathbb{1} - \hat{U}^*\hat{U}) - \hat{\mathcal{T}}_\xi(\mathbb{1} - \hat{U}\hat{U}^*) = \hat{\mathcal{T}}_\xi\text{−}\mathrm{Ind}(\hat{U}).$$

Since $\hat{U} - \mathbf{P}u_F\mathbf{P} \in \mathcal{K}(\mathcal{N})$ due to the splitting one concludes

$$\hat{\mathcal{T}}_\xi\text{−}\mathrm{Ind}(\hat{U}) = \hat{\mathcal{T}}_\xi\text{−}\mathrm{Ind}(\mathbf{P}u_F\mathbf{P}) = \mathrm{Ch}_{\mathcal{T},\xi}(u_F)$$

by the Index theorem 2.3.5 and the invariance properties of the Fredholm index. □

This proof of bulk-boundary correspondence works because the halfspace projection is precisely the spectral projection of a Dirac operator and no analogous approach will work for different codimensions. The idea as well as the strategy of proof for the splitting condition in the pseudogapped case originate from the Master thesis of the author [118], which was subsequently improved and extended to the mobility gapped case in [111].

To prove the splitting condition (6.1.1) we employ again resolvent functional calculus, i.e. approximate the sign-function with the holomorphic functions of Lemma A.6. To compare with the bulk we use the geometric resolvent identity

$$\frac{\mathbf{P}}{F(\hat{H}) - z\mathbf{P}} = \mathbf{P}\frac{1}{F(H) - z}\mathbf{P} + \frac{\mathbf{P}}{F(\hat{H}) - z\mathbf{P}}\hat{V}\frac{1}{F(H) - z}\mathbf{P} \tag{6.1.2}$$

with $\frac{\mathbf{P}}{F(\hat{H})-z\mathbf{P}}$ denoting inverses in the algebra $\mathcal{N}$ and with the perturbation

$$\hat{V} = \mathbf{P}F(H)(\mathbb{1} - \mathbf{P}) + \hat{v} \in L^p(\mathcal{N})$$

where $\hat{v}$ is as in Assumption 6.1.1. The first term of (6.1.2) integrates to $\mathbf{P}\mathrm{sgn}(H)\mathbf{P}$ for a contour chosen as in Lemma A.6 and the strategy is to show that the contour integral of the second term is bounded in some L^p-(quasi)-norm, which then implies immediately that it is an element of $\mathcal{K}(\mathcal{N})$.

Here it is important that $F(H)$ is a matrix over the unitization of some Sobolev space since otherwise $\hat{V}$ could not be compact. If that assumption where to be dropped one would have to do a relative bulk-boundary correspondence, i.e. one would need to compare two halfspace models and hope to cancel all terms which are not compact. It is conceivable that one can deal with that situation, but the main problem would be that there is no obvious reason to expect that $\operatorname{sgn}(\hat{H}_1) - \operatorname{sgn}(\hat{H}_2)$ is in any way smoother than an arbitrary spectral projection.

Our criteria for the splitting conditions are basically the same as in [111] derived for tight-binding models even the main ideas from the proofs adapt almost verbatim. The first condition, which is mostly for exhibition of the strategy derives the splitting from a pseudogap of large order:

Proposition 6.1.3 (cf. [111, Proposition 5.6.5]) *If the Hamiltonians $H, \hat{H}$ are as in Assumption 6.1.1 and H has a pseudogap of order $\gamma > 2$ at 0 then the splitting property (6.1.1) property is satisfied with*

$$\operatorname{sgn}(\hat{H}) - \mathbf{P}\operatorname{sgn}(H)\mathbf{P} \in L^p(\mathcal{N})$$

for any $2 \leq p < \gamma$.

Proof. As stated above, we want to apply

$$\begin{aligned}\operatorname{sgn}(\hat{H}) - \mathbf{P}\operatorname{sgn}(H)\mathbf{P} &= \operatorname*{s-lim}_{\epsilon\to 0}(\operatorname{sgn}_\epsilon(\hat{H}) - \mathbf{P}\operatorname{sgn}_\epsilon(H)\mathbf{P}) \\ &= \operatorname*{s-lim}_{\epsilon\to 0}\int_{\mathcal{C}_\epsilon} \frac{\mathbf{P}}{F(\hat{H}) - z\mathbf{P}} - \mathbf{P}\frac{1}{F(H)-z}\mathbf{P}\mathrm{d}z\end{aligned}$$

with notations as in Lemma A.6. Choose a finite interval $\Delta = [-M, M]$ around 0 where the density of states of H is well-defined and a nonnegative function φ supported in Δ and equal to 1 on $[-\frac{1}{2}M, \frac{1}{2}M]$. Separating the bulk resolvent as

$$(F(H)-z)^{-1} = (F(H)-z)^{-1}\varphi(H) + (F(H)-z)^{-1}(1-\varphi(H))$$

we obtain

$$\begin{aligned}
&\frac{\mathbf{P}}{F(\hat{H})-z\mathbf{P}}-\mathbf{P}\frac{1}{F(H)-z}\mathbf{P}\\
&=\frac{\mathbf{P}}{F(\hat{H})-z\mathbf{P}}\hat{V}\frac{1}{F(H)-z}(1-\varphi(H))\mathbf{P}+\frac{\mathbf{P}}{F(\hat{H})-z\mathbf{P}}\hat{V}\frac{1}{F(H)-z}\varphi(H)\mathbf{P}\\
&=\frac{\mathbf{P}}{F(\hat{H})-z\mathbf{P}}\hat{V}\frac{1}{F(H)-z}(1-\varphi(H))\mathbf{P}+\frac{\mathbf{P}}{F(\hat{H})-z\mathbf{P}}\hat{V}\frac{1}{F(H)}\varphi(H)\mathbf{P}\\
&+\frac{\mathbf{P}}{F(\hat{H})-z\mathbf{P}}\hat{V}\left(\frac{1}{F(H)-z}-\frac{1}{F(H)}\right)\varphi(H)\mathbf{P}
\end{aligned}$$

where the inverse of $F(H)$ exists on the range of $\varphi(H)$ by Proposition 4.3.17. The first line of the final expression is rather easily seen to integrate to something bounded in L^p-norm, after all $\hat{V}\in L^p(\mathcal{N})$ by assumption and

$$\int_{\mathcal{C}_\epsilon}\frac{\mathbf{P}}{F(\hat{H})-z\mathbf{P}}\hat{V}\frac{1}{F(H)}\varphi(H)\mathbf{P}\mathrm{d}z=(2\pi\imath)\mathbf{P}\mathrm{sgn}_\epsilon(\hat{H})\hat{V}\frac{1}{F(H)}\varphi(H)\mathbf{P}$$

as well as

$$\begin{aligned}
\int_{\mathcal{C}_\epsilon}\frac{\mathbf{P}}{F(\hat{H})-z\mathbf{P}}\hat{V}\frac{1-\varphi(H)}{F(H)-z}\mathbf{P}\mathrm{d}z&=(2\pi\imath)\mathbf{P}\mathrm{sgn}_\epsilon(\hat{H})\hat{V}\frac{1-\varphi(H)}{F(H)}\mathbf{P}\\
&+\int_{\mathcal{C}_\epsilon}\frac{\mathbf{P}}{F(\hat{H})-z\mathbf{P}}\hat{V}\left(\frac{1}{F(H)-z}-\frac{1}{F(H)}\right)(1-\varphi(H))\mathbf{P}\mathrm{d}z
\end{aligned}$$

where the remaining integral is uniformly norm-bounded in ϵ since the singular part around $z\to 0$ is compensated by

$$\left\|\left(\frac{1}{F(H)-z}-\frac{1}{F(H)}\right)(1-\varphi(H))\right\|=O(|z|).$$

To bound the remaining part we sum $\hat{V}=\sum_{l\in\mathbb{N}}\hat{V}\mathbf{P}_l$ which converges in the norm of $L^\infty(\mathcal{N})$ by Assumption 6.1.1. The final part is bounded via

$$\begin{aligned}
&\int_{\mathcal{C}_\epsilon}\left\|\frac{\mathbf{P}}{F(\hat{H})-z\mathbf{P}}\hat{V}\mathbf{P}_l\left(\frac{1}{F(H)-z}-\frac{1}{F(H)}\right)\varphi(H)\mathbf{P}\right\|_p\mathrm{d}z\\
&\leq\int_{\mathcal{C}_\epsilon}|\Im m z|^{-1}\left\|\hat{V}\mathbf{P}_l\right\|\left\|\mathbf{P}_l\left(\frac{1}{F(H)-z}-\frac{1}{F(H)}\right)\varphi(H)\right\|_p\mathrm{d}z
\end{aligned}$$

$$\leq (2L)^{1/p} \int_{\mathcal{C}_\epsilon} |\Im m z|^{-1} \|\hat{V}\mathbf{P}_l\| \left\| \left(\frac{1}{F(H) - z} - \frac{1}{F(H)} \right) \varphi(H) \right\|_{L^p(\mathcal{A})} \mathrm{d}z$$

$$\leq c(2L)^{1/p} \|\hat{V}\mathbf{P}_l\| \int_{\mathcal{C}_\epsilon} |\Im m z|^{-1+s} \mathrm{d}z$$

for finite $0 < s \leq 1$ where we used Proposition 2.1.5 to get to the third line and then Proposition 4.3.17. The final integral is bounded uniformly in ϵ since the singularity at the real line is integrable. □

In one of the most important cases, namely that of two-dimensional Hamiltonians with Dirac-points one only has pseudogap of order less than 2, hence one must use a different criterion. The proof above must be modified since Proposition 2.1.5 does not hold for small exponents. One therefore needs to use that the resolvents also have a small amount of fractional smoothness:

Proposition 6.1.4 ([111, Proposition 5.6.6]) *Let $H, \hat{H}$ be as in Assumption 6.1.1. Assume that for some smooth compactly supported function φ equal to 1 on a neighborhood of 0 the norm*

$$\sup_{z \in D_{M,\kappa}} \left\| \frac{1}{F(H) + z} \varphi(H) \right\|_{B^{\frac{1}{2}}_{1,1}(\mathcal{A})} < \infty$$

is bounded on some cone as in Proposition 4.3.17 and one has an estimate

$$\left\| \left(\frac{1}{F(H) - z} - \frac{1}{F(H)} \right) \varphi(H) \right\|_{B^{\frac{1}{2}}_{11}(\mathcal{A})} \leq c\, |\Im m z|^s$$

for some $0 < s < 1$ and $z \in D_{M,\kappa}$. Then the splitting property (6.1.1) holds with

$$\operatorname{sgn}(\hat{H}) - \mathbf{P}\operatorname{sgn}(H)\mathbf{P} \in L^p(\mathcal{N})$$

for each $p_0 < p < \infty$.

Proof. We describe how to modify the proof of Proposition 6.1.3 to obtain the result. Separating off the part $K_1(\epsilon)$ coming from $(1 - \varphi(H))$ one has two norm-bounded SOT-convergent sequences such that

$$\operatorname{sgn}(\hat{H}) - \mathbf{P}\operatorname{sgn}(H)\mathbf{P} = \operatorname*{s-lim}_{\epsilon \to 0} K_1(\epsilon) + (2\pi\imath)^{-1} \operatorname*{s-lim}_{\epsilon \to 0} K_2(\epsilon)$$

with

$$K_1(\epsilon) = \int_{C_\epsilon} \frac{\mathbf{P}}{F(\hat{H}) - z\mathbf{P}} \hat{V} \frac{1}{F(H)} \varphi(H)\mathbf{P}\mathrm{d}z = (2\pi\imath)\mathbf{P}\mathrm{sgn}_\epsilon(\hat{H})\hat{V}\frac{1}{F(H)}\varphi(H)\mathbf{P}$$

defining an element of $L^p(\mathcal{N})$ due to the pseudogap. Clearly $K_1(\epsilon)$ has uniformly bounded operator norm and L^p-norm. The remainder

$$K_2(\epsilon) =$$
$$\int_{C_\epsilon} \left(\frac{\mathbf{P}}{F(\hat{H}) - z\mathbf{P}} \hat{V} \frac{\varphi(H)}{F(H)} \mathbf{P} + \frac{\mathbf{P}}{F(\hat{H}) - z\mathbf{P}} \hat{V} \left(\frac{1}{F(H) - z} - \frac{1}{F(H)} \right) \varphi(H)\mathbf{P} \right) \mathrm{d}z$$

therefore also has operator norm finite and bounded uniformly in ϵ.

It is sufficient to demonstrate that $\sup_{\epsilon>0} \|K_2(\epsilon)\|_1$ is finite and this can be done precisely as in the proof except that one applies Proposition 2.1.8 instead of Proposition 2.1.5, e.g.

$$\left\| P_\ell \left(\frac{1}{F(H) - z} - \frac{1}{F(H)} \right) \varphi(H) \right\|_1 \leq c \left\| \left(\frac{1}{F(H) - z} - \frac{1}{F(H)} \right) \varphi(H) \right\|_{B^{\frac{1}{2}}_{1,1}} .$$

Since also $\sup_{\epsilon>0} \|K_2(\epsilon)\| < \infty$ one has $\sup_{\epsilon>0} \|K_2(\epsilon)\|_p < \infty$ by interpolation. □

The complicated sufficient condition actually holds automatically if one has a pseudogap of order larger $\gamma > \frac{3}{2}$ which includes the case of Dirac points in two dimensions:

Proposition 6.1.5 *Let H be as in Assumption 6.1.1 and further has a pseudogap of order $\gamma > 1$ at $E = 0$ then*

(i) $\frac{1}{F(H)}\varphi(H) \in B^s_{1,1}(\mathcal{M})$ *for each* $0 < s < \gamma - 1$.

(ii) For any $0 < s < \gamma - 1$ there exists some $0 < \tilde{s} < 1$ for which there is a constant $C > 0$ such that

$$\left\| \left(\frac{1}{F(H) - z} - \frac{1}{F(H)} \right) \varphi(H) \right\|_{B^s_{1,1}} \leq C\,|\epsilon|^{\tilde{s}} + \mathcal{O}(\epsilon) \qquad (6.1.3)$$

for ϵ small enough.

Proof. There is not much to add to the rather technical argument given in [111, Proposition 5.6.7] which generalizes straightforwardly to the situation here. The gist is that one decomposes $\varphi = \sum_{k=0}^{\infty} \varphi_k$ into dyadic parts supported on intervals $(-2^{-k}, -2^{-k-1}) \cup (2^{-k-1}, 2^{-k})$ and estimates the scaling behavior of $\left\| \frac{1}{F(H)} \varphi_k(H) \right\|_1$ and $\left\| \nabla \frac{1}{F(H)} \varphi_k(H) \right\|_1$ in k that one gets from the pseudogap. Those norms behave as $O(2^{-(\gamma-1)k})$ and as $O(2^{-(\gamma-2)k})$ respectively which can be combined to yield the fractional smoothness that is needed here. □

In the disordered regime we can use the fractional-moments bound instead of a pseudogap to control the size matrix elements of the resolvent in the quasi-norm of $L^q(\mathcal{N})$ for $0 < q < 1$.

Lemma 6.1.6 *Assume that H as in Assumption 6.1.1 has a γ-Hölder continous DOS and a mobility gap in an interval Δ containing 0.*

Fix some $\delta > 0$. For any $0 < q < s < \gamma$ there are constants c_1, c_2, c_3 such that

$$\left\| \mathbf{P}_l \frac{1}{F(H) - z_1} \varphi(H) \right\|_q \leq c_1 , \tag{6.1.4}$$

$$\left\| \mathbf{P}_l \left(\frac{1}{F(H) - z_1} - \frac{1}{F(H) - z_2} \right) \varphi(H) \right\|_q \leq c_2 \, |z_1 - z_2| \, \left|\Im m(z_1)\right|^{-\frac{q}{s}} \tag{6.1.5}$$

$$\left\| \mathbf{P}_l \frac{1}{(F(H) - z)^2} \varphi(H) \right\|_q \leq c_3 \left|\Im m(z_1)\right|^{-\frac{q}{s}} \tag{6.1.6}$$

uniformly in $l \in \mathbb{Z}$ and z_1, z_2 with $\operatorname{dist}(z_i, \sigma(h) \setminus \Delta) > \delta$.

Proof. We recall from Proposition 4.3.14 that H also satisfies a fractional moments bound as in Definition 4.3.11. One must note that the action θ there does not have to be the full $\mathbb{R}^d$-action, we can also use only ξ. Thus we also have a fractional moments bound over $L^\infty(\mathcal{A}) \rtimes_\xi G$, i.e. for any exponent $0 < s < \gamma$ there are constants

$$\left\| \mathbf{P}_{r_1} \frac{1}{F(H) + z_1} \varphi(H) \mathbf{P}_{r_2} \right\|_s^s \leq A_{s,k} \langle r_1 - r_2 \rangle^{-k}$$

for z_1 as in the statement of the Lemma. The first estimate (6.1.4) is then straightforward, one just sums over a partition of unity

$$\left\|\mathbf{P}_r \frac{1}{F(H)-z_1}\varphi(H)\right\|_q^q \leq \sum_{r_2\in\mathbb{Z}} \left\|\mathbf{P}_r P_x \frac{1}{F(H)-z_1}\varphi(H)\mathbf{P}_{r_2}\right\|_q^q.$$

The other estimates follow once one knows how to bound the products of two resolvents, so we demonstrate one such case. One expands

$$\mathbf{P}_r \frac{1}{F(H)-z_1}\frac{1}{F(H)-z_2}\varphi(H) = \sum_{r_1,r_2\in\mathbb{Z}} \mathbf{P}_r \frac{\tilde{\varphi}(H)}{F(H)-z_1}\mathbf{P}_{r_1}\frac{\varphi(H)}{F(H)-z_2}\mathbf{P}_{r_2}$$

using another cutoff supported on a slightly larger interval with $\varphi = \tilde{\varphi}\varphi$. Since a combination of the Hölder inequality and log-convexity gives

$$\|ab\|_q \leq \|a\|_{\frac{qs}{s-q}} \|b\|_s \leq \|a\|_q^{1-\frac{q}{s}} \|a\|_\infty^{\frac{q}{s}} \|b\|_s$$

we can estimate

$$\begin{aligned}&\left\|\sum_{r_1,r_2\in\mathbb{Z}} \mathbf{P}_r \frac{\tilde{\varphi}(H)}{F(H)-z_1}\mathbf{P}_{r_1}\frac{\varphi(H)}{F(H)-z_2}\mathbf{P}_{r_2}\right\|_q^q \\ &\leq \sum_{r_1,r_2\in\mathbb{Z}}\left(|\Im m z_1|^{1-\frac{q}{s}} \left\|\mathbf{P}_r \frac{\tilde{\varphi}(H)}{F(H)-z_1}\mathbf{P}_{r_1}\right\|_q^{1-\frac{q}{s}} \left\|\mathbf{P}_{r_1}\frac{\varphi(H)}{F(H)-z_2}\mathbf{P}_{r_2}\right\|_s\right)^q\end{aligned}$$

which brings it into a form where the fractional moments bound can be applied. □

In the following we will need to estimate the L^p-quasinorm of certain contour integrals. One must therefore be aware of some facts regarding integration in p-Banach spaces. While the definitions of the Riemann integral adapts without changes not all continuous functions are integrable and also there is no obvious triangle inequality available to estimate the size of an integral even if it converges. For analytic functions that have series expansions $f(z) = \sum_{n=0}^{\infty} f_n(z)x_n$ with scalar functions f_n, however, the Riemann integrals exist and are equal to their termwise integrals [61]

$$\int_\Gamma f(z)\,\mathrm{d}z = \sum_{n=0}^{\infty} x_n \int_\Gamma f_n(z)\,\mathrm{d}z\,,$$

thus they can be bounded as

$$\left\| \int_\Gamma f(z)\,\mathrm{d}z \right\|^p \leq \sum_{n=0}^{\infty} \|x_n\|^p\, \|f_n\|_\infty^p\, |\Gamma|^p\,. \tag{6.1.7}$$

This case covers everything that is needed already.

Proposition 6.1.7 ([111, Proposition 5.6.9]) *Let $H, \hat{H}$ be as in Assumption 6.1.1 and let H have a γ-Hölder continous DOS and a mobility gap in an interval Δ containing 0. Then*

$$\operatorname{sgn}(\hat{H}) - \mathbf{P}\operatorname{sgn}(H)\,\mathbf{P} \in \mathcal{K}(\mathcal{N})\,.$$

Proof. As argued in the proof of Proposition 6.1.4 one can split off a part converging in some norm $L^p(\mathcal{N})$ and only has to prove that

$$K_2(\epsilon) = \int_{C_\epsilon} \left(\frac{\mathbf{P}}{F(\hat{H}) - z\mathbf{P}} \hat{V} \frac{1}{F(H)} \varphi(H)\mathbf{P} + \frac{\mathbf{P}}{F(\hat{H}) - z\mathbf{P}} \hat{V} \left(\frac{1}{F(H) - z} - \frac{1}{F(H)} \right) \varphi(H)\mathbf{P} \right) \mathrm{d}z$$

has uniformly bounded $L^q(\mathcal{N})$-quasinorm. We can even prove that this net converges in the q-quasinorm, thus its strong limit is in particular compact. Summing again $\hat{V} = \sum_l \hat{V} P_l$ one part of $K_2(\epsilon)$ is

$$\frac{1}{2\pi} \int_{C_\epsilon} \frac{\mathbf{P}}{F(\hat{H}) - z} \hat{V} \mathbf{P}_l \frac{1}{F(H)} \varphi(H)\mathbf{P}\mathrm{d}z = (2\pi\imath)\operatorname{sgn}_\epsilon(\hat{H}) \hat{V} \mathbf{P}_l \frac{1}{F(H)} \varphi(H)\mathbf{P}.$$

This term converges to $\operatorname{sgn}(\hat{H})\hat{V}\mathbf{P}_l \frac{1}{F(H)} \varphi(H)\mathbf{P} \in L^q(\mathcal{N})$ in the q-quasinorm by Lemma 1.3.1 since $\mathbf{P}_l \varphi(H) \frac{1}{F(H)} \in L^q(\mathcal{N})$ holds by Lemma 6.1.6 and the other factor converges in SOT.

The more difficult to estimate contributions to $K_2(\epsilon)$ are of the form

$$\int_{C_\epsilon} \frac{\mathbf{P}}{F(\hat{H}) - z\mathbf{P}} \hat{V} \mathbf{P}_l \left(\frac{1}{F(H) - z} - \frac{1}{F(H)} \right) \varphi(H) \mathrm{d}z$$

where two factors depend on z. To estimate this contour integral we must use the theory of integration in quasi-Banach spaces as sketched above. Thus one has to expand the integrand as a power series in terms of z and integrate term-wise.

It is clear that outer rectangle of the contour can be estimated rather easily since the only problematic parts are those where the contour approaches the spectrum of $F(\hat{H})$. Due to symmetry it is further enough if we demonstrate how to bound the part of the contour with positive imaginary part, i.e. the integral $\int_\epsilon^1 G_m(z)\, \mathrm{d}z$ with integrand

$$G_m(z) \;=\; \frac{\mathbf{P}}{F(\hat{H}) - z\mathbf{P}} \hat{V} \mathbf{P}_l \left(\frac{1}{F(H) - z} - \frac{1}{F(H)} \right) \varphi(H) \mathbf{P}.$$

The region of integration will be split into dyadic intervals $I_j = (z_{j+1}, z_j)$ for $\epsilon = 2^{-m}$, $0 \le j < m$ where we expand around the points $z_j = 2^{-j}$. In I_j the q-norm-convergent series expansion is

$$G_m(z) \;=\; \sum_{n=0}^{\infty} x_{L,n}^{(j)}\, g_n^{(j)}(z)\;,$$

with $g_n^{(j)}(z) = -(\imath)^n (z - z_j)^n$ and

$$\begin{aligned}
x_{L,n}^{(j)} \;&=\; \sum_{k=0}^{n} \frac{1}{(F(H) - \imath z_j)^{k+1}}\, \hat{V}\, \mathbf{P}_l \left(\frac{1}{(F(H) - \imath z_j)^{n-k+1}} \right) \varphi(H) \mathbf{P} \\
&\quad - \frac{\mathbf{P}}{(F(\hat{H}) - \imath z_j \mathbf{P})^{n+1}}\, \hat{V} \mathbf{P}_l\, \frac{1}{F(H) - \imath z_j} \varphi(H)\, \mathbf{P} \\
&= \sum_{k=0}^{n-1} \frac{1}{(F(H) - \imath z_j)^{k+1}}\, \hat{V}\, \mathbf{P}_l \left(\frac{1}{(F(H) - \imath z_j)^{n-k+1}} \right) \varphi(H) \mathbf{P} \\
&\quad - \frac{\mathbf{P}}{(F(\hat{H}) - \imath z_j \mathbf{P})^{n+1}}\, \hat{V}\, \mathbf{P}_l \left(\frac{1}{F(H)} - \frac{1}{F(H) - \imath z_j} \right) \varphi(H)\, \mathbf{P}
\end{aligned}$$

In the second line each term of the sum contains the factor $\mathbf{P}_l \frac{1}{F(H)-\imath z_j)^2}\varphi(H)$ that can be estimated with (6.1.6) and in the third line one can apply (6.1.5). Bounding all other resolvents in each term with the standard resolvent estimate $\left\|\frac{1}{F(H)-\imath z}\right\|_\infty \le \frac{1}{|\Im m z|}$ one has

$$\begin{aligned}\left\|x_{L,n}^{(j)}\right\|_q^q &\le \left(c_1 \sum_{k=0}^{n-1} \left|z_j\right|^{-nq-q\frac{q}{s}} + c_2 \left|z_j\right|^{-(n+1)q-q\frac{q}{s}} \left|z_j\right|^q\right) \left\|\hat{V}\mathbf{P}_l\right\|^q \, \left\|\mathbf{P}_l\right\|_q^q \\ &\le (c_3 n + c_4) 2^{jq(n+\frac{q}{s})} \left\|\hat{V}\mathbf{P}_l\right\|^q \,.\end{aligned}$$

uniformly in n, j, l and L. Together with the trivial bound

$$\left\|g_n^{(j)}\right\|_\infty = \sup_{z\in[z_{j+1},z_j]} \left|g_n^{(j)}(z)\right| \le 2^{-(j+1)n}$$

the termwise integral (6.1.7) thus gives

$$\begin{aligned}\left\|\int_{z_{j+1}}^{z_j} G_L(z)\,\mathrm{d}z\right\|_q^q &\le \sum_{n=0}^{\infty} (c_3 n + c_4) 2^{jq(n+\frac{q}{s})} 2^{-q(j+1)n} \left|z_j - z_{j-1}\right|^q \\ &= \sum_{n=0}^{\infty} (c_3 n + c_4) 2^{jq(n+\frac{q}{s})} 2^{-q(j+1)n} \, 2^{-(j+1)q} \\ &= \sum_{n=0}^{\infty} (c_3 n + c_4) 2^{-qn+qj(\frac{q}{s}-1)-q} \\ &= c_5 2^{jq(\frac{q}{s}-1)} \,,\end{aligned}$$

such that $q < s$ implies that taking the sum over j gives a uniform upper bound for the original integral. In fact, one therefore has convergence in the q-quasi-norm for $\epsilon \to 0$ due to dominated convergence. In the end we conclude that

$$\mathbf{P}\mathrm{sgn}(\hat{H})\mathbf{P} - \mathbf{P}\mathrm{sgn}(H_+)\mathbf{P} \in (L^p(\mathcal{N}) + L^q(\mathcal{N})) \cap \mathcal{N} \subset \mathcal{N} \cap L^p(\mathcal{N})$$

is compact. □

For completeness we formulate the main results of this section as a Theorem:

Theorem 6.1.8 *let H, $\hat{H}$ be chiral Hamiltonians that satisfy Assumption 6.1.1 and either of the following*

(i) H has a pseudogap at 0 of order $\gamma > \frac{3}{2}$.

(ii) H has a spectral gap or mobility gap in an interval Δ containing 0.

Then

$$\hat{\mathcal{T}}_{\xi}(J\mathrm{Proj}_{\mathrm{Ker}(\hat{H})}) = Ch_{\mathcal{T},\xi}(u_F),$$

in particular if the bulk winding number does not vanish then $\hat{H}$ has a non-trivial kernel.

Proof. Just combine Proposition 6.1.2 with the correct splitting condition: In the pseudogapped case Proposition 6.1.3 for $\gamma > 2$, the combination of Proposition 6.1.4 and Proposition 6.1.5 and in the mobility gapped case Proposition 6.1.7. □

Note also that in the mobility gapped case and for a suitable pseudogap the part of Assumption 6.1.1 regarding regularity of the Fermi unitary may hold automatically due to the results of Section 4.3. Similar to the algebraic bulk-boundary correspondence this result is robust in the sense that one can modify $\hat{H}$ by a largely arbitrary edge term which will not destroy the exact 0-energy modes (as long as the chiral symmetry is preserved). The result is still a statement about elements of a certain operator algebra and one must in the end relate it to a physical representation, where we recall that a Hamiltonian in the algebraic sense often corresponds to a whole family of physical Hamiltonians labeled e.g. by a space disorder configurations and possibly other parameters as well. In [111, Section 5.2] the example of ergodic tight-binding models on a halfspace is worked out and it is shown that for those one has in fact almost surely a non-trivial kernel also in the physical representation. In the case of a weak Chern number, i.e. in dimensions larger than one, the kernel must moreover be infinitely degenerate. In models that are translation-invariant in the directions orthogonal to the boundary, this kernel appears as a flat bands of edge modes and can also be observed in nature, e.g. as the 0-energy modes of graphene with zigzag edges [89, 42].

6.2 Interface currents

We start with a similar algebraic setup as Section 6.1, namely starting from an observable algebra $(\mathcal{A}, \theta, \mathcal{T})$ with a 1-parameter-restriction ξ of θ generated by a normal vector $v \in S^{d-1}$. We consider here interfaces instead of halfspaces, but apart from that we are again interested in bulk Hamiltonians with mobility or pseudogaps. As in Section 5.4 we have an underlying exact sequence

$$0 \to \mathcal{E} \to \hat{\mathcal{A}} \to \mathcal{A} \oplus \mathcal{A} \to 0$$

where $\mathcal{E} = \mathcal{A} \rtimes_\xi G_v$, $\hat{\mathcal{A}} := \mathrm{T}(\mathcal{A}, G_v, \xi)$ is the two-sided Toeplitz extension for the action ξ as in Section 6.1 spanned by a unit vector v and where G_v can be a torus if ξ is periodic. We assume that $L^\infty(\mathcal{E})$ is faithfully represented on some Hilbert space $\mathcal{H}$, and that the representation is covariant w.r.t. θ, thus the action is generated by commuting self-adjoint operators $X_1, \ldots, X_d$. The generator X_ξ is equal to $v \cdot X$. Except for the construction of the algebra $L^\infty(\mathcal{E})$ the exact sequence plays little role in this section, it only motivates the results in the case of spectrally gapped bulk Hamiltonians.

In this setup we have a pair of bulk Hamiltonians H_+, H_- which we will assume for simplicity to be strongly affiliated to $\mathcal{A}$ and which are joined by an interface Hamiltonian $\hat{H}_I$ which is also strongly affiliated to $\hat{\mathcal{A}}$. Furthermore, the differences between $\hat{H}_i$ and $H_\pm$ should be bounded operators. More concretely we are thinking of a situation

$$\hat{H}_I = H_0 + \frac{1}{2}\{V_+, \mathcal{P}_+\} + \frac{1}{2}\{V_-, \mathcal{P}_-\} + \hat{v}$$

where H_0 is a strongly 2-smooth reference Hamiltonian, $V_\pm \in M(\mathcal{A})$ are smooth potentials, $\mathcal{P}_\pm = f_\pm(X_\xi)$ are smooth switch functions of the (abstract) generator of the crossed product $L^\infty(\mathcal{E}) = L^\infty(\mathcal{A}) \rtimes_\xi G_v$ and $\hat{v}$ is a term that is localized to the interface (i.e. vanishes at infinity). A simple example that can be written in this form is a Schrödinger-type Hamiltonian

$$\hat{H}_I = -\nabla^2 + V_- \mathcal{P}_- + V_+ \mathcal{P}_+$$

which interpolates between two periodic potentials $V_\pm$. Another example in two dimensions that can be written in this form and also satisfies all other assumptions of this section is a domain-wall configuration of regularized two-dimensional Dirac-Hamiltonians

$$\hat{H}_I = \begin{pmatrix} m_- + \epsilon\nabla^2 & \imath\nabla_x + \nabla_y \\ \imath\nabla_x - \nabla_y & -m_- - \epsilon\nabla^2 \end{pmatrix} + \mathcal{P}_+ \begin{pmatrix} m_+ - m_- & 0 \\ 0 & -(m_+ - m_-) \end{pmatrix}$$

which interpolates between two different mass terms $m_\pm$.

The physical picture is that the bulk Chern numbers of the Fermi projections

$$\mathrm{Ch}_{w\times v}(e_\pm) := \langle \mathrm{Ch}_{\mathcal{T},w\times v}, [e_\pm]_0 - [s(e_\pm)]_0\rangle$$

evaluated for two orthogonal directions w, v are proportional to the transverse Hall conductances for the respective directions. They are typically integer-valued in two dimension. If the difference of Chern numbers is non-vanishing across the interface then there should arise gap-filling states localized to the interface region but which can propagate in the plane of the interface orthogonal to the normal vector v. Their chiral velocity in the direction w is a topological invariant equal to the difference of bulk Chern number (and therefore quantized in two dimensions).

If H_-, H_+ have a common bulk gap Δ and g is a smooth function supported in Δ one can define a weighted interface current as

$$\sigma^I(w, g, \hat{H}_I) := -\imath\hat{\mathcal{T}}_\xi(g(\hat{H}_I)[\hat{H}_I, X_w])$$

which is nothing but the expectation value of the velocity operator $[\hat{H}_I, X_w]$ for a direction $w \in S^{d-1}$, $w \perp v$ in the state $g(\hat{H}_I) \in L^1(\mathcal{N})$. This is well-defined since functions of $\hat{H}_I$ which are supported in the bulk gap are properly localized to the interface region. For normalization one may assume that g is non-negative and has unit integral. One can compute directly [74, 103, 78]) that the current is proportional to the pairing $\langle \mathrm{Ch}_{\hat{\mathcal{T}}_\xi,w}, [\hat{u}_\Delta]_1\rangle$ with the boundary unitary under the exact sequence of Section 5.4, to be precise

$$\begin{aligned}\sigma^I(w, g, \hat{H}_I) &= \frac{1}{2\pi}\langle \mathrm{Ch}_{\hat{\mathcal{T}}_\xi,w}, [\hat{u}_\Delta]_1\rangle \\ &= \frac{1}{2\pi}\langle \mathrm{Ch}_{\mathcal{T},w\times v}, [\chi(H_+ < E_F)]_0^M - [\chi(H_- < E_F)]_0^M\rangle.\end{aligned}$$

Without the bulk gaps this definition of a surface current does not make sense anymore since $g(\hat{H}_I)$ is always also supported in the bulk. Nevertheless, if the bulk is insulating (i.e. if $H_\pm$ have mobility gaps in Δ) then the expectation would be that large-scale transport can only happen in the interface region. To define an interface current one can try to define regularizations that at first take only a finite strip around the interface region into account but then leverage cancellations in the bulk to give rise to a finite limit as the regulator is removed [40, 49, 51, 119, 123]. Such regularizations are non-unique and in absence of any better motivation, e.g. from response theory, we will choose the one which is the most suitable to recover a bulk-interface respectively bulk-boundary correspondence in a certain limit. For that reason we follow the regularization given by [51] for two-dimensional tight-binding Hamiltonians and adapt it to a sensible abstract version that also applies to higher-dimensional systems with non-trivial two-dimensional weak Chern numbers.

We first choose a smooth cutoff in real space. In the following denote for consistency $X_v = X_\xi$ the generator of the crossed product. As a partition of space introduce a partition of unity $\mathbb{1} = \sum_{m\in\mathbb{Z}} \phi(X_v - m)$ for some non-negative smooth function ϕ compactly supported in the interval $(-\frac{3}{2}, \frac{3}{2})$. Then we denote

$$\Pi_L = \sum_{m=-L}^{L} \phi(X_v - m), \qquad \overline{\Pi}_L = \mathbb{1} - \Pi_L$$

and further decompose $\overline{\Pi}_{L\pm} = \overline{\Pi}_L \chi(0 < \pm X_\xi)$, i.e. $\overline{\Pi}_{L\pm} = \sum_{m=L+1}^{\infty} \phi(X_\xi \mp m)$ as a strongly convergent sum.

The regularized edge current is defined as the trace of

$$\sigma_L^I(w, T, g, \hat{H}_I) := \frac{1}{T}\int_0^T \frac{\imath}{2}\hat{\mathcal{T}}_\xi\left(\{[\hat{H}_I, X_w], \Pi_L^{(t)}\}g(\hat{H}_I)\right)\mathrm{d}t$$

where $\Pi_L = \varphi(\frac{1}{L}X_\xi)$ and $\Pi_L^{(t)} = e^{\imath t F(\hat{H}_I)}\Pi_L e^{-\imath t F(\hat{H}_I)}$ with F a smooth strictly monotonous function which behaves like $\operatorname{sgn}(\lambda) + O(\lambda^{-2})$ at infinity. For simplicity one can take the bounded transform though there is a large freedom. For bounded H one can take $F = \mathrm{id}$ and recovers the regularization from [51] in two dimensions. The trace here can be justified to exist under the assumptions given below even though $[\hat{H}_I, X_w]$ is generally unbounded and can be simplified using the operator

$$\Sigma_L(w, t, g; \hat{H}_I) := \Pi_L^{(t)}[\hat{H}_I, X_w]g(\hat{H}_I)$$

which is related to the previous regularization via

$$\sigma_L^I(w, T, g; \hat{H}_I) = \frac{1}{T}\int_0^T \Im m \hat{\mathcal{T}}_\xi(\Sigma_L(w, t, g; \hat{H}_I))\mathrm{d}t$$

since the left-hand side is real-valued and

$$\Im m\left(g^{\frac{1}{2}}(\hat{H}_I)\Pi_L^{(t)}[\hat{H}_I, X_w]g^{\frac{1}{2}}(\hat{H}_I)\right) = \frac{1}{2}g^{\frac{1}{2}}(\hat{H}_I)\{\Pi_L^{(t)}, [\hat{H}_I, X_w]\}g^{\frac{1}{2}}(\hat{H}_I).$$

The regularization can be motivated in different ways and a physical interpretation is given already in [51], let us therefore focus on two points in particular. The first point is the averaging over the dynamics of $F(\hat{H}_I)$. First of all, we take a bounded function of $\hat{H}_I$ since it gives much-needed analytic properties to the exponential $e^{\imath F(\hat{H}_I)t}$, which is especially important in our abstract situation where the true dynamics $e^{\imath \hat{H}_I t}$ is difficult to estimate using only smoothness of $\hat{H}_I$. Morally speaking, it should not make much of a difference if one averages over $\hat{H}_I$ or a function of $\hat{H}_I$ which is almost constant outside Δ since one has the factor $g(\hat{H}_I)$ which also provides some localization to a finite energy region. In the end, the time-average will filter out certain contributions to the interface current which average to 0 under the dynamics in the bulk.

The other point is the localization of the velocity operator to a strip which is given by the self-adjoint operator

$$\frac{\imath}{2}\{f(X_\xi), [\hat{H}_I, X_w]\} = \frac{\imath}{2}(f(X_\xi)[\hat{H}_I, X_w] + [\hat{H}_I, X_w]f(X_\xi))$$

with f a compactly supported function. One can think of different ways to restrict an observable to a finite region, but this choice here is natural and preferred since it is affine in f as is expected for an observable quantity that grows proportionally to the system size. Another good property for the regularized interface current to have is that it should be equal to 0 identically if $H_+ = H_-$ and that is indeed the case for this choice, but would not necessarily be true for other localizations.

We can now state the precise technical assumptions that are imposed on the Hamiltonians:

Assumption 6.2.1 *Let $d = 2$ or $d = 3$ and let H_+, H_- be strongly p-smooth Hamiltonians for all $p \in (\frac{d}{2}, \infty]$ and in addition for both $\sigma \in \{-, +\}$*

(i) H_σ is $(X, \frac{1}{4})$-smooth in the sense of Definition 1.4.11, i.e. among other things $[H_\sigma, X_v](H_\sigma^2 + 1)^{-\frac{1}{4}}$ extends to a bounded operator.

(ii) There is a reference Hamiltonian H_0, which is a self-adjoint $\mathcal{A}$-multiplier with a a spectral gap in Δ, and such that $H_\sigma = H_0 + V_\sigma$ for some bounded p-smooth perturbations.

(iii) $(H_\sigma + \imath)^{-1}$ lies in some $L^p(\mathcal{A})$ for $p \in (\frac{d}{2}, \infty]$ for $d = 2$ or $d = 3$. If $d = 3$ then H_σ must in addition be bounded from below.

(iv) *If Θ is a switch function with compactly supported derivative Θ' then $H_\sigma, \hat{H}_I$ should be smooth w.r.t. $\Theta(X_\xi)$ with $\eta = \frac{1}{2}$, i.e. among other things the operators $[\Theta(X_\xi), H_\sigma](H_\sigma^2+1)^{-\frac{1}{4}-\epsilon}$ extend to bounded operators. In addition there should be for each $\epsilon > 0$ and $j \in \mathbb{N}$ a constant $C_{\epsilon,j}$ such that*

$$\left\| P_x[\Theta(X_\xi), H_\sigma](H_\sigma^2 + 1)^{-\frac{1}{4}-\epsilon} P_y \right\| \leq C_{j,\epsilon} \langle x - y\rangle^{-j} \langle x\rangle^{-j} \langle y\rangle^{-j}.$$

(v) *For each $j \in \mathbb{N}$ there is a constant C_j such that*

$$\left\| P_x(\hat{H}_I - H_\sigma)P_y \right\| \leq C_j \langle x - y\rangle^{-j} \langle x_d\rangle^{-j}$$

for all $m \in \mathbb{Z}$ with $\operatorname{sgn}(x_d) = \sigma$.

Here the matrix elements refer to the projections $P_x = \chi(X \in x + [0,1))^d$ as in Section 4.3. In addition to those assumption one will need a mobility gap or pseudogap. The conditions are geared towards the situation

$$\hat{H}_I = H_0 + \frac{1}{2}\{V_+, \mathcal{P}_+\} + \frac{1}{2}\{V_-, \mathcal{P}_-\}$$

where H_0 is a second order (matrix-valued) differential operator in low spatial dimensions and $V_\pm$ is a bounded matrix-valued potential (constant, periodic or ergodic). We assume the strong affiliation to make sense of the Chern numbers of the bulk Fermi projections. One could drop that condition and still obtain a concise expression for the interface current but we are not aware of any model Hamiltonian that satisfies all of the other assumptions and is not strongly affiliated. The potentials $V_\pm$ and also the additional interface term are assumed to be bounded operators for simplicity; with some additional effort one could probably extend the results to also allow them to be relatively bounded, e.g. first-order differential operators if H_0 is second-order.

The assumptions also cover the discrete case. Indeed, for short-range tight-binding models based on a unital algebra the conditions are mostly trivial since the resolvents and even the Hamiltonians H_σ themselves are trace-class. While d is here used as the dimension of space it only acts as a proxy for the assumed L^p-regularity; for tight-binding models everything applies in higher dimension as well (after all one can just restrict θ to the action generated by w and v). The assumption do not cover the substrate models of Section 4.3.5 since those are never resolvent- or strongly affiliated. Analogues of the results of this section are probably true for those as well, at least they hold almost verbatim in the spectrally gapped case where they can be derived using K-theory. One would, however, need to find a different starting point for the proof. Let us also note that, despite the notations, X_w need not be a position operator for some spatial direction, it may as well be a directional derivative w.r.t. some internal degrees of freedom or a parameter.

We comment further on limitations and possible extensions in Section 6.2.5.

6.2.1 Results and strategy

Our result in the mobility gap regime is:

Theorem 6.2.2 *Let $\hat{H}_I$, H_σ be as in the general assumptions but assume in addition that H_+ and H_- have a common mobility gap Δ. If g is a C^∞-function properly supported on the interior of Δ with $\int_\Delta g(\lambda)\mathrm{d}\lambda = 1$ then*

$$\lim_{T\to\infty}\lim_{L\to\infty}\sigma_L^I(w,T,g;\hat{H}_I) = \lim_{T\to\infty}\lim_{L\to\infty}\frac{1}{T}\int_0^T \Im m\hat{\mathcal{T}}_\xi\left(\Sigma_L(w,t,g;\hat{H}_I)\right)\mathrm{d}t$$
$$= \frac{1}{2\pi}\int_\Delta g(\lambda)\left(Ch_{v\times w}(\chi(H_+<\lambda)) - Ch_{v\times w}(\chi(H_-<\lambda))\right)\mathrm{d}\lambda$$

In particular if the Chern numbers are constant in the gap (as is expected in certain situations) then the right-hand side is simply the difference of the bulk Chern numbers.

We now present the strategy of the proof. Since it is quite long we give the main intermediate results without interruption and give the proofs in the following sections.

For technical reasons we include a spectral cutoff and set

$$\Sigma_L(\varphi,w,t,g;\hat{H}_I) := \varphi(\hat{H}_I)\Pi_L^{(t)}[\hat{H}_I,X_w]g(\hat{H}_I)\varphi(\hat{H}_I)$$

for any smooth (often compactly supported) function φ. Whenever $\varphi g = g$ then the regularized edge current is equal to the time average of

$$\Im m\hat{\mathcal{T}}_\xi(\Sigma_L(\varphi,w,t,g;\hat{H}_I) = \hat{\mathcal{T}}_\xi(\varphi(\hat{H}_I)\Pi_L^{(t)}[\hat{H}_I,X_w]g(\hat{H}_I))$$

since φ drops out using cyclicity.

We now set up the functional calculus. With the antiderivative $G(\lambda) = 1 - \int_{-\infty}^{\lambda} g(x)\mathrm{d}\lambda$ the smooth functional calculus (see Appendix A) allows to write

$$
\begin{aligned}
G(\hat{H}_I) &= \frac{1}{2\pi}\int_{\mathbb{C}} (\partial_{\overline{z}} G_K)(z)\frac{1}{\hat{H}_I - z}\mathrm{d}z \wedge \mathrm{d}\overline{z},\\
g(\hat{H}_I) &= \frac{1}{2\pi}\int_{\mathbb{C}} (\partial_{\overline{z}} G_K)(z)\frac{1}{(\hat{H}_I - z)^2}\mathrm{d}z \wedge \mathrm{d}\overline{z}.
\end{aligned}
$$

where both integrals can be restricted the set $D = \{0 < \langle \Im m z\rangle < 2\langle \Re e z\rangle\} \subset \mathbb{C}$ (the first integral has to be understood in the strong operator topology [51], the second converges in operator norm). Moreover, the construction of G_K shows that on the real line $\partial_{\overline{z}} G_K$ can only be non-vanishing in $\mathrm{supp}(G') = \mathrm{supp}(g)$. If $\hat{H}_I$ is bounded from below one may further replace G with a function that vanishes below the spectrum and then the integrals can be restricted to a fixed compact subset of $\mathbb{C}$. In the following we will use the abbreviation $\mathrm{d}\nu(z) := (\partial_{\overline{z}} G_K)(z)\mathrm{d}z \wedge \mathrm{d}\overline{z}$ where we assert that K can always be chosen large enough to ensure convergence of all expressions.

A key point of the bulk-boundary correspondence (as identified by [51]) is that

$$
[G(\hat{H}_I), X_w] = \frac{1}{2\pi}\int_{\mathbb{C}} \frac{1}{\hat{H}_I - z}[\hat{H}_I, X_w]\frac{1}{\hat{H}_I - z}\mathrm{d}\nu(z) \tag{6.2.1}
$$

$$
[\hat{H}_I, X_w] g(\hat{H}_I) = \frac{1}{2\pi}\int_{\mathbb{C}} [\hat{H}_I, X_w]\frac{1}{(\hat{H}_I - z)^2}\mathrm{d}\nu(z) \tag{6.2.2}
$$

can be related to each under a trace using cyclicity.

The first step is to show convergence of $\hat{\mathcal{T}}_\xi(\Sigma_L(\varphi, w, t, g; \hat{H}_I))$ in the limit $L \to \infty$. Following [51] this is facilitated by adding and subtracting an additional term

$$
Z_L(\varphi, w, t, g; \hat{H}_I) = \varphi(\hat{H}_I)\Pi_L[G(\hat{H}_I), X_w]\varphi(\hat{H}_I) \tag{6.2.3}
$$

$$
+ \frac{1}{2\pi}\int_D \varphi(\hat{H}_I)\left(\frac{1}{\hat{H}_I - z}\Pi_L^{(t)}[\hat{H}_I, X_w] - \Pi_L^{(t)}[\hat{H}_I, X_w]\frac{1}{\hat{H}_I - z}\right)\frac{1}{\hat{H}_I - z}\varphi(\hat{H}_I)\mathrm{d}\nu(z) \tag{6.2.4}
$$

for which the L^1-norm of $\Sigma_L(w, t, g; \hat{H}_I) + Z_L(w, t, g; \hat{H}_I)$ becomes bounded uniformly in L. Both lines are $\hat{\mathcal{T}}_\xi$-trace-class since Π_L provides localization to the interface region. The trace of the second line (6.2.4) vanishes by cyclicity and we will later show that the same is true for the first line in the limit $\varphi \to 1$ where the regulator is removed. Hence the additional term does not contribute to the trace eventually.

Using

$$\frac{1}{2\pi}\int_D \Pi_L^{(t)}\frac{1}{\hat H_I - z}[\hat H_I, X_w]\frac{1}{\hat H_I - z}\mathrm{d}\nu(z) \\ = \frac{-1}{2\pi}\int_D \Pi_L^{(t)}\left[\frac{1}{\hat H_I - z}, X_w\right]\mathrm{d}\nu(z) = -\Pi_L^{(t)}[G(\hat H_I), X_w]$$

one can rewrite

$$\Sigma_L(\varphi, w, t, g; \hat H_I) + Z_L(\varphi, w, t, g; \hat H_I) \\ = \varphi(\hat H_I)\Pi_L[G(\hat H_I), X_w]\varphi(\hat H_I) + \frac{1}{2\pi}\int_D \varphi(\hat H_I)\frac{1}{\hat H_I - z}\Pi_L^{(t)}[\hat H_I, X_w]\frac{1}{\hat H_I - z}\varphi(\hat H_I)\mathrm{d}\nu(z) \\ = \varphi(\hat H_I)(\Pi_L - \Pi_L^{(t)})[G(\hat H_I), X_w]\varphi(\hat H_I) \\ + \frac{1}{2\pi}\int_D \varphi(\hat H_I)\frac{1}{\hat H_I - z}[\Pi_L^{(t)}, \hat H_I]\frac{1}{\hat H_I - z}[\hat H_I, X_w]\,\frac{1}{\hat H_I - z}\varphi(\hat H_I)\mathrm{d}\nu(z).$$

Going over to the complement $\overline{\Pi}_L = \mathbb{1} - \Pi_L = \overline{\Pi}_{L+} + \overline{\Pi}_{L-}$, again with $\overline{\Pi}_{L\sigma}^{(t)} = e^{\imath \hat H_I t}\overline{\Pi}_{L\sigma}e^{-\imath \hat H_I t}$ this becomes

$$\Sigma_L(\varphi, w, t, g; \hat H_I) + Z_L(\varphi, w, t, g; \hat H_I) \\ = \sum_{\sigma\in\{-,+\}} \varphi(\hat H_I)(\overline{\Pi}_{L\sigma}^{(t)} - \overline{\Pi}_{L\sigma})[G(\hat H_I), X_w]\varphi(\hat H_I) \tag{6.2.5}$$

$$- \frac{1}{2\pi}\int_D \varphi(\hat H_I)\frac{1}{\hat H_I - z}[\overline{\Pi}_{L\sigma}^{(t)}, \hat H_I]\frac{1}{\hat H_I - z}[\hat H_I, X_w]\,\frac{1}{\hat H_I - z}\varphi(\hat H_I)\mathrm{d}\nu(z). \tag{6.2.6}$$

The point of this switch-around is that $\overline{\Pi}_{L\sigma}^{(t)} - \overline{\Pi}_{L\sigma}$ and the commutators are well-localized around the two lines $v \cdot x = \pm L$ which recede more and more into the bulk as L goes to infinity. Thus one can safely take the limit:

Lemma 6.2.3 *We have*

$$\lim_{L\to\infty} \hat{\mathcal{T}}_\xi\left(\Sigma_L(\varphi, w, t, g; \hat H_I) + Z_L(w, t, g; \hat H_I)\right) = \sum_{\sigma\in\{-,+\}} \hat{\mathcal{T}}_\xi(\Sigma_\sigma(\varphi, w, t, g; H_\sigma)) \\ + \hat{\mathcal{T}}_\xi(Z_\infty(\varphi, w, t, g; \hat H_I)$$

with

$$\Sigma_\sigma(\varphi, w, t, g; H_\sigma) = \varphi(H_\sigma)(\overline{\Pi}_{0\sigma} - \overline{\Pi}_{0\sigma}^{(t,\sigma)})[G(H_\sigma), X_w]\varphi(H_\sigma) + \frac{1}{2\pi}\int_D \varphi(H_\sigma)\frac{1}{H_\sigma - z}[\overline{\Pi}_{0\sigma}^{(t,\sigma)}, H_\sigma]\frac{1}{H_\sigma - z}[H_\sigma, X_w]\,\frac{1}{H_\sigma - z}\varphi(H_\sigma)\mathrm{d}\nu(z). \tag{6.2.7}$$

where $\overline{\Pi}_{0\sigma}^{(t,\sigma)} = e^{\imath H_\sigma t}\overline{\Pi}_{0\sigma}e^{-\imath H_\sigma t}$ *and one has the error term*

$$\hat{\mathcal{T}}_\xi(Z_\infty(\varphi, w, t, g; \hat{H}_I)) = \lim_{L\to\infty}\hat{\mathcal{T}}_\xi(Z_L(\varphi, w, t, g; \hat{H}_I))$$
$$= \sum_{\sigma\in\{-,+\}}\hat{\mathcal{T}}_\xi(\varphi(\hat{H}_I)\overline{\Pi}_{0\sigma}[G(\hat{H}_I), X_w]\varphi(\hat{H}_I) - \varphi(H_\sigma)\overline{\Pi}_{0\sigma}[G(H_\sigma), X_w]\varphi(H_\sigma))$$

We call the contribution of Z_∞ an error term since it vanishes in the limit where the regulator φ is removed:

Lemma 6.2.4 *For the scaling limit $\varphi_\epsilon = \varphi(\epsilon\cdot)$ one has*

$$\lim_{\epsilon\to 0}\hat{\mathcal{T}}_\xi(Z_\infty(\varphi_\epsilon, w, t, g; \hat{H}_I)) = 0.$$

The remaining expression, save for the error term, only involves the two bulk Hamiltonians. Using a general formula one can rewrite it in terms of the bulk trace by exchanging a commutator with the switch function $\overline{\Pi}$ with a derivative in direction X_v:

Lemma 6.2.5

$$\hat{\mathcal{T}}_\xi(\Sigma_\sigma(\varphi, w, t, g; H_\sigma))$$
$$= \sigma\mathcal{T}\left(\varphi(H_\sigma)e^{\imath F(H_\sigma)t}[X_d, e^{-\imath F(H_\sigma)t}]\,[G(H_\sigma), X_w]\varphi(H_\sigma)\right)$$
$$- \sigma\frac{1}{2\pi}\int_D \mathcal{T}\left(R(\varphi, w, t, g, z; H_\sigma)\right)\mathrm{d}\nu(z).$$

with

$$R(\varphi, w, t, g, z; H_\sigma) :=$$
$$\varphi(H_\sigma)\frac{1}{H_\sigma - z}e^{\imath F(H_\sigma)t}[X_d, H_\sigma]e^{-\imath F(H_\sigma)t}\frac{1}{H_\sigma - z}[H_\sigma, X_w]\,\frac{1}{H_\sigma - z}\varphi(H_\sigma)$$

Then we take the time average over t which makes the remainder term vanish identically:

Lemma 6.2.6

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_D \mathcal{T}(R(\varphi,w,t,g,z;H_\sigma))\mathrm{d}\nu(z)=0$$

The remaining term decomposes into the average of Chern numbers across the mobility gap:

Lemma 6.2.7 *With $e_{\sigma\lambda}=\chi(H_\sigma<\lambda)$ one has*

$$\lim_{\varphi\to\mathbb{1}}\lim_{T\to\infty}\frac{1}{T}\int_0^T\mathcal{T}(\Sigma_\sigma(\varphi,w,t,g;H_\sigma))=-\frac{1}{2\pi\imath}\int_{\mathbb{R}}g(\lambda)Ch_{v\times w}(e_{\sigma\lambda})\mathrm{d}\lambda.$$

Since the right-hand side is purely imaginary one immediately reads off the limit of the surface current which completes the proof.

We now come to the pseudogapped case. Here one needs to use a different set of assumptions, namely that H_σ both have a sublinear pseudogap at some E_F (corresponding to linear band-touching points in three dimensions) and are bounded from below. The second assumption is necessary for a specific technical reason, it allows us to use efficient contour integrals for the window function g. For that reason we use a specific family of window functions g_β that are holomorphic instead of compactly supported. They are derived from a holomorphic approximation to the Fermi projection, namely we set $g_\beta=-G_\beta'$ for the Fermi-Dirac distribution function

$$G_\beta(\lambda)=(1+e^{\beta(\lambda-E_F)})^{-1}$$

and inverse temperature $\beta>0$. In the limit $\beta\to\infty$ one recovers the Fermi projection with the SOT-limit $\text{s-lim}_{\beta\to\infty}G_\beta(H_\sigma)=\chi(H_\sigma<E_F)$ since that is the pointwise limit of G_β and E_F is not an eigenvalue of H_σ. Despite appearances, this choice has no immediate physical motivation but was rather made for the ease of setting up the functional calculus. Using the scaling limit of a more general switch function, e.g. one with compactly supported derivative, seems possible, however, we would rather not expend that much effort on the extremely technical estimates that would be necessary to bound the integrals from the smooth functional calculus with the pseudogap bound of Proposition 4.3.17.

As a contour for the functional calculus we employ the boundary of the region

$$R_\beta = \{z \in \mathbb{C} : \left|z - \sigma(\hat{H}_I)\right| \leq 1 \ \wedge \ \frac{|\Im m\, z|}{\langle \beta(\Re e z - E_F)\rangle} < \frac{1}{\beta}\}$$

which surrounds the spectrum of $\hat{H}_I$ (and thus of $H_\pm$) while keeping a distance of 1 except for a wedge in the vicinity of E_F to avoid enclosing the singularities of G_β at $\imath\frac{\pi}{\beta}(1 + 2\mathbb{Z})$. Since the distance to the singularities scales with β it is easy to see that $\sup_{\beta>0} \sup_{z\in\partial R_\beta} \left|G_\beta(z)\right| < \infty$, i.e. the functions are uniformly bounded on the contours.

The functional calculus is then given by the two norm-convergent contour integrals

$$G_\beta(\hat{H}_I) = \frac{1}{2\pi} \int_{\partial R_\beta} G_\beta(z) \frac{1}{\hat{H}_I - z} \mathrm{d}z, \quad g_\beta(\hat{H}_I) = \frac{1}{2\pi} \int_{\partial R_\beta} G_\beta(z) \frac{1}{(\hat{H}_I - z)^2} \mathrm{d}z.$$

We abbreviate $d\nu_\beta(z) := G_\beta(z)\mathrm{d}z$ in the following.

Theorem 6.2.8 *In addition to Assumption 6.2.1 let*

(i) H_σ both have a pseudogap at a fixed E_F with order $\gamma > 2$,

(ii) H_σ both be bounded from below.

Then

$$\lim_{T\to\infty} \lim_{\beta\downarrow 0} \lim_{L\to\infty} \sigma_L^I(w, T, g_\beta; \hat{H}_I) = \lim_{T\to\infty} \lim_{\beta\downarrow 0} \lim_{L\to\infty} \frac{1}{T} \int_0^T \Im m \hat{\mathcal{T}}_\xi \left(\Sigma_L(w, t, g_\beta; \hat{H}_I)\right) \mathrm{d}t$$
$$= \frac{1}{2\pi} \left(Ch_{v\times w}(\chi(H_+ < E_F)) - Ch_{v\times w}(\chi(H_- < E_F))\right).$$

Compared to the mobility gap regime we must take the limit where g_β converges to a δ-function. Let us also note that the bulk is typically a conductor for a semimetallic Hamiltonian, therefore the regularization not only provides meaning to the interface current, it also has to filters out contributions by the delocalized bulk states. This is done by taking the limit $\beta \to 0$ which we would have to take anyhow to have a precise bulk-boundary correspondence, since the bulk Chern number may only be well-defined at the single energy E_F. For a discussion of similar points also concerning the conductivity at finite temperature see also [123].

We again give the proof strategy (where it is assumed in the following Lemmas that the conditions of Theorem 6.2.8 hold). The $L \to \infty$ limit is taken similarly to the mobility gapped case:

Lemma 6.2.9 *We have*

$$\lim_{L\to\infty}\lim_{\epsilon\to 0}\hat{\mathcal{T}}_\xi\left(\Sigma_L(\varphi(\epsilon\cdot),w,t,g;\hat{H}_I)+Z(\varphi(\epsilon\cdot),w,t,g_\beta;\hat{H}_I)\right)$$
$$=\sum_{\sigma\in\{-,+\}}\sigma\hat{\mathcal{T}}_\xi(\Sigma_\sigma(w,t,g_\beta;H_\sigma))$$

with

$$\Sigma_\sigma(w,t,g;H_\sigma)=(\overline{\Pi}_{0+}-\overline{\Pi}_{0+}^{(t,\sigma)})[G_\beta(H_\sigma),X_w] + \frac{1}{2\pi}\int_D \frac{1}{H_\sigma-z}[\overline{\Pi}_{0+}^{(t,\sigma)},H_\sigma]\frac{1}{H_\sigma-z}[H_\sigma,X_w]\frac{1}{H_\sigma-z}\mathrm{d}\nu_\beta(z). \tag{6.2.8}$$

where $\overline{\Pi}_{0\sigma}^{(t,\sigma)}=e^{\imath F(H_\sigma)t}\overline{\Pi}_{0\sigma}e^{-\imath F(H_\sigma)t}$.

The cutoff must be removed immediately here since g_β is not compactly supported. As the Hamiltonians are bounded from below, G_β is rapidly decaying when restricted to $\sigma(\hat{H}_I$ and so $G_\beta(\hat{H}_I)$ is well-localized to the interface regions, which means that all expressions are well-defined without a regulator.

This bulk expression is almost the same as the one of Lemma 6.2.3 such that the starting point is the $\beta\to\infty$-limit of

$$\hat{\Sigma}_\sigma(T,g_\beta):=\frac{1}{T}\int_0^T\left(\mathcal{T}\left(e^{\imath F(H_\sigma)t}[X_d,e^{-\imath F(H_\sigma)t}]\,[G_\beta(H_\sigma),X_w]\right)\mathrm{d}t\right. \tag{6.2.9}$$
$$\left.+\frac{1}{2\pi}\int_{D_\beta}\mathcal{T}\left(\frac{1}{H_\sigma-z}e^{\imath F(H_\sigma)t}[X_d,H_\sigma]e^{-\imath F(H_\sigma)t}\frac{1}{H_\sigma-z}[H_\sigma,X_w]\,\frac{1}{H_\sigma-z}\right)\mathrm{d}\nu_\beta(z)\right)\mathrm{d}t$$

for $g_\epsilon=-G_\epsilon$.

Similar to the mobility gapped case the first term converges to the Chern number in the limit, while the second vanishes.

Lemma 6.2.10

$$\lim_{T\to\infty}\lim_{\beta\to\infty}\frac{1}{T}\int_0^T \mathcal{T}\left(e^{\imath F(H_\sigma)t}[X_d, e^{-\imath F(H_\sigma)t}]\,[G_\beta(H_\sigma), X_w]\right) = \frac{1}{2\pi\imath}Ch_{v\times w}(e_F).$$

Lemma 6.2.11

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_{D_\beta}\mathcal{T}\left(\frac{e^{\imath F(H_\sigma)t}}{H_\sigma - z}[X_d, H_\sigma]\frac{e^{-\imath F(H_\sigma)t}}{H_\sigma - z}[H_\sigma, X_w]\,\frac{1}{H_\sigma - z}\right)\mathrm{d}\nu_\beta(z)\mathrm{d}t = 0.$$

6.2.2 Preliminaries

We begin with some preliminary notions and results to better characterize the regularity of various operators as they appear in the following.

Definition 6.2.12 *For $a \in L^\infty(\mathcal{E})$ and $p \in [1,\infty]$ define the seminorms*

$$\|a\|_{\mathcal{S}_{p,N}} := \sup_{x,y\in\mathbb{Z}^d}\langle x-y\rangle^N \left\|P_x\, aP_y\right\|_p$$

and

$$\|a\|_{\mathcal{L}_{p,N}} := \sup_{x,y\in\mathbb{Z}^d}\langle x-y\rangle^N\langle y\rangle^N \left\|P_x\, aP_y\right\|_p.$$

Denote by $\mathcal{S}_p$ respectively $\mathcal{L}_p$ the set of all $a \in L^\infty(\mathcal{E})$ such that $\|a\|_{\mathcal{S}_{p,N}} < \infty$ respectively $\|a\|_{\mathcal{L}_{p,N}} < \infty$ for all $N \in N$. The former operators are called p-smooth and the latter are called p-localized.

If there is a family $(a_i)_{i\in I}$ of operators for which each of the respective seminorms is bounded uniformly (i.e. independent of i) then we say that the family is uniformly p-smooth respectively p-localized.

This notion of localization has little relation to dynamic localization; what is meant is simply that the matrix elements decay rapidly with the distance to the interface region. The linear spaces $\mathcal{S}_p$ and $\mathcal{L}_p$ are locally convex spaces with their respective seminorms and at least for $\mathcal{L}_p$ it is easy to take the completion to a Fréchet space which embeds into $L^p(\mathcal{E})$ (this is, however, not necessary for our purposes). There are some simple algebraic relations between those classes of operators, e.g. there are jointly continuous products $\mathcal{S}_p \times \mathcal{S}_q \to \mathcal{S}_r$ and $\mathcal{L}_p \times \mathcal{S}_q \to \mathcal{L}_r$ for Hölder-dual exponents $r^{-1} = p^{-1} + q^{-1}$.

We can now state some basic results on the regularity of various expressions:

Lemma 6.2.13 *Let σ be $+$ or $-$, $\mathcal{P}_+ = \Theta(X_\xi)$ a smooth restriction to the positive halfspace as in Section 3 and $\mathcal{P}_- = \mathbb{1} - \mathcal{P}_+$.*

(i) The resolvents $(\hat{H}_I + z)^{-1}$ and $(H_\sigma + z)^{-1}$ are p-smooth for any $p \in (\frac{d}{2}, \infty]$, the bounded operators $[\hat{H}_I, X_w](\hat{H}_I + z)^{-1}$ and $[H_\sigma, X_w](H_\sigma + z)^{-1}$ are p-smooth for $p \in (d, \infty]$ and any $z \in \mathbb{C} \setminus \mathbb{R}$.

(ii) For ϕ a Schwartz function, $\phi(\hat{H}_I)$ and $\phi(H_\sigma)$ are p-smooth and the difference $\phi(\hat{H}_I) - \mathcal{P}_+\phi(H_+) - \mathcal{P}_-\phi(H_-)$ is p-localized for each $p \in [1, \infty]$.

(iii) For G a switch function with $G(\lambda) = O(1) + O(|\lambda|^{-2})$, $G(\hat{H}_I)$ and $G(H_\sigma)$ are ∞-smooth and the difference $g(\hat{H}_I) - \mathcal{P}_+G(H_+) - \mathcal{P}_-G(H_-)$ is p-localized for each $p \in (\frac{d}{2}, \infty]$.

(vi) The map $a \mapsto [\mathcal{P}_+, a]$ is a bounded (continuous) map from $\mathcal{S}_p$ to $\mathcal{L}_p$.

(v) For φ a Schwartz function denote its family of dilations by $\varphi_\epsilon = \varphi(\epsilon\cdot)$. Then each $\varphi_\epsilon(H_\sigma)$, $\epsilon > 0$, is p-smooth for $p \in (d, \infty]$ and

$$\lim_{\epsilon \to 0} \left\| \nabla \varphi_\epsilon(H_\sigma) \right\|_{\mathcal{S}_{p,N}} = 0$$

for each $N \in \mathbb{N}$.

Each $\varphi_\epsilon(\hat{H}_I) - \mathcal{P}_+\varphi_\epsilon(H_+) - \mathcal{P}_-\varphi_\epsilon(H_-)$, $\epsilon > 0$ is p-localized for $p \in (\frac{d}{2}, \infty]$ and

$$\lim_{\epsilon \to 0} \left\| \varphi_\epsilon(\hat{H}_I) - \mathcal{P}_+\varphi_\epsilon(H_+) - \mathcal{P}_-\varphi_\epsilon(H_-) \right\|_{\mathcal{L}_{p,N}} = 0$$

for each $N \in \mathbb{N}$.

Proof.
(i): We know that $(\hat{H}_I + z)^{-1}$ and $(H_\sigma + z)^{-1}$ are ∞-smooth and locally in L^p for $p \in (\frac{d}{2}, \infty]$ i.e. $\left\| P_x(\hat{H}_I + z)^{-1}P_y \right\|_p < \infty$ for any such p. By interpolation that implies p-smoothness for the same range. Similarly $[\hat{H}_I, X_w](\hat{H}_I^2 + 1)^{-\frac{1}{4}-\epsilon}$ and $[H_\sigma, X_w](H_\sigma^2 + 1)^{-\frac{1}{4}-\epsilon}(H_\sigma + z)^{-1}$ are ∞-smooth for any $\epsilon > 0$ by Lemma 1.4.15. Also, $(\hat{H}_I^2 + 1)^{\frac{1}{4}-\epsilon}(\hat{H}_I + \imath)^{-1}$ is locally L^p for any $p \in [\frac{d}{1-4\epsilon}, \infty)$ from which one obtains $[H_\sigma, X_w](H_\sigma + \imath)^{-1} \in L^p(\mathcal{A})$ for the same range of exponents p and similarly for the $L^p(\mathcal{E})$-regularity of $P_x[\hat{H}_I, X_w](\hat{H}_I + z)^{-1}P_y$.

(ii) and (iii): Due to Assumption 6.2.1(i) and Proposition 1.4.13 one has ∞-smoothness of $f(H_\sigma)$ or $F(\hat{H}_\sigma)$ for any function $f \in \bigcap_{\beta>-1} \mathcal{S}^\beta(\mathbb{R})$. For any Schwartz function one has in addition for $K > \frac{d}{2}$ a constant such that

$$\sup_{x,y} \left\|P_x f(H_\sigma) P_y\right\|_1 < c_{f,K} \sup_{x,y} \left\|P_x (1+H_\sigma^2)^{-\frac{K}{2}} P_y\right\|_1 < \infty,$$

which implies p-smoothness for any $1 \leq p < \infty$ through interpolation. For the differences we must estimate the matrix elements more precisely. From Proposition 4.3.13 we have

$$\sup_{x,y\in\mathbb{Z}^d} \left\|P_x (H_\sigma + z)^{-1} P_y\right\| \langle x-y\rangle^K \leq C_K \sum_{m=0}^{K+d} |\Im m\, z|^{-1} \left(1 + \frac{\langle \Re e\, z\rangle}{|\Im m\, z|}\right)^m$$

and similarly for $\hat{H}_I$. Together with

$$\sup_{x,y\in\mathbb{Z}^d} \left\|P_x (H_\sigma + \imath)^{-1} P_y\right\|_p \leq \left(1 + \frac{\langle \Re e\, z\rangle}{|\Im m\, z|}\right) \left\|(H_\sigma+\imath)^{-1}\right\|_{L^p(\mathcal{A})}$$

for any $p \in (\frac{d}{2}, \infty]$ we have using log-convexity (1.3.2)

$$\begin{aligned}
&\left\|P_x (H_\sigma + \imath)^{-1} P_y\right\|_q \\
&\leq c_{p,q} \langle x-y\rangle^{\theta K} C_K^\theta\, |\Im m\, z|^{-\theta} \left(1 + \frac{\langle \Re e\, z\rangle}{|\Im m\, z|}\right)^{(K+d)\theta + (1-\theta)} \left\|(H_\sigma+\imath)^{-1}\right\|_{L^p(\mathcal{A})}^{1-\theta}
\end{aligned}$$

for any $q > p$ and $0 < \theta < 1$ with $\frac{1}{q} = \frac{1-\theta}{p}$. For $\operatorname{sgn}(x \cdot v) = \sigma$ and any $q > p > \frac{d}{2}$ we therefore estimate with the resolvent identity

$$\begin{aligned}
&\left\|P_x (\hat{H}_I - z)^{-1} P_y - P_x (H_\sigma - z)^{-1} P_y\right\|_q \\
&\leq \sum_{w_1, w_2 \in \mathbb{Z}^d} \left\|P_x (\hat{H}_I - z)^{-1} P_{w_1}\right\| \left\|P_{w_1} (V_\sigma - \hat{V}) P_{w_2}\right\| \left\|P_{w_2} (H_\sigma - z)^{-1} P_y\right\|_q \\
&\leq C\, |\Im m\, z|^{-1-\theta} \left(1 + \frac{\langle \Re e\, z\rangle}{|\Im m z|}\right)^{c(K,\theta,q)} \\
&\qquad \sum_{w_1, w_2 \in \mathbb{Z}^d} \langle x - w_1\rangle^{-N} \langle w_1 - w_2{}^{-K} \langle w_1 \cdot v\rangle^{-K} \langle w_2 - y\rangle^{-K}
\end{aligned}$$

with arbitrarily large K where we inserted our Assumption 6.2.1(v) about $V_\sigma - \hat{V} = H_\sigma - \hat{H}_I$ with $c(K,\theta,q)$ some large exponent. Combining the decays one sees that the resolvent-difference is q-localized. Due to the smooth functional calculus the difference $f(\hat{H}_I) - \mathcal{P}_+ f(H_+) - \mathcal{P}_- f(H_-)$ for any function $f \in \mathcal{S}^{-1-\theta}(\mathbb{R})$ is q-smooth and if θ can be arbitrarily small (such as for the switch function G) then this extends to all $q \in (\frac{d}{2}, \infty)$.

(iv): For suitable functions $\Theta_\pm$ with $\Theta_+\Theta_- = 0$ and compactly supported functions f_1, f_2, g_1, g_2 one can write

$$[\mathcal{P}_+, a] = \Theta_+(X_v)a\Theta_-(X_v) - \Theta_-(X_v)a\Theta_+(X_v) + f_1(X_v)a + af_2(X_v) + g_1(X_v)ag_2(X_v)$$

and it is not difficult to show that each of these terms is p-localized in terms of the p-smoothness of a.

(v): We can write using the smooth functional calculus

$$\nabla\varphi(\epsilon H_\sigma) = \int_{\mathbb{C}} (\partial_{\overline{z}}\tilde{\varphi}_K)(z)(\epsilon H_\sigma - z)^{-1}\epsilon(\nabla H_\sigma)(\epsilon H_\sigma - z)^{-1} \mathrm{d}z \wedge \mathrm{d}\overline{z}.$$

The resolvents are semiuniformly smooth in the sense that

$$\left\|(\epsilon H_\sigma - z)^{-1}\right\|_{\mathcal{S}_{\infty,N}} \leq c_N \left|\Im m\, z\right|^{-c_1(N)} \left(1 + \frac{\langle \Re e\, z\rangle}{|\Im m z|}\right)^{c_2(N)} |||\epsilon H|||_{m_1(N),\frac{1}{4}}$$

and

$$\left\|\nabla(\epsilon H_\sigma - z)^{-1}\right\|_{\mathcal{S}_{\infty,N}} \leq c_N \left|\Im m\, z\right|^{-c_3(N)} \left(1 + \frac{\langle \Re e\, z\rangle}{|\Im m z|}\right)^{c_4(N)} |||\epsilon H|||_{m_2(N)+1,\frac{1}{4}}$$

for powers $c_1(N), ..., c_4(N)$ and integers $m_1(N), m_2(N)$ depending on N and the commutators are estimated using the expressions from Definition 1.4.11 for which we note the scaling

$$|||\epsilon H|||_{N,\frac{1}{4}} = \sup_{|j|=N} \left\|\epsilon(\nabla^j H)(1+\epsilon^2 H^2)^{-\frac{1}{4}}\right\| \leq \epsilon^{\frac{1}{2}} |||H|||_{N,\frac{1}{4}}.$$

Since φ is compactly supported this means

$$\left\|\nabla\varphi(\epsilon H_\sigma)\right\|_{\mathcal{S}_{\infty,N}} \leq c_N \epsilon^{\frac{1}{2}}$$

for constants independent of ϵ. Applying log-convexity in the form

$$\left\|P_x a P_y\right\|_q \leq \left\|P_x a P_y\right\|_p^{\frac{q}{p}} \left\|P_x a P_y\right\|_\infty^{1-\frac{q}{p}}$$

for any $q > p$ we need only find out how the local L^p-norm $\nabla(\epsilon H_\sigma - z)^{-1}$ behave for $\epsilon \to 0$. We begin with the estimate

$$\begin{aligned}&\left\|P_x \nabla(\epsilon H_\sigma - z)^{-1} P_y\right\|_p \\ &\leq c_N \left|\Im m\, z\right|^{-c_1(N)} \left(1 + \frac{\langle \Re e\, z\rangle}{|\Im m z|}\right)^{c_2(N)} \left\|(\epsilon H_\sigma + \imath)^{-1} \nabla(\epsilon H_\sigma)(\epsilon H_\sigma + \imath)^{-1}\right\|_{L^p(\mathcal{A})}\end{aligned}$$

and the first two factors are unproblematic since they are compensated in the smooth functional calculus formula by the almost analytic extension; the more delicate factor is estimated as

$$\begin{aligned}&\left\|(\epsilon H_\sigma + \imath)^{-1} \nabla(\epsilon H_\sigma)(\epsilon H_\sigma + \imath)^{-1}\right\|_{L^p(\mathcal{A})} \\ &\leq \left\|(1 + \epsilon^2 H_\sigma^2)^{-\frac{1}{4}}\right\|_p \left\|(1 + \epsilon^2 H_\sigma^2)^{-\frac{1}{4}} \nabla(\epsilon H_\sigma)(\epsilon H_\sigma + \imath)^{-1}\right\|_\infty .\end{aligned}$$

One sees that the second factor grows with ϵ as $O(\epsilon^{\frac{1}{2}})$ while the first decreases

$$\left\|(1 + \epsilon^2 H_\sigma^2)^{-\frac{1}{4}}\right\|_p = \epsilon^{-\frac{1}{2}} \left\|(\epsilon^{-2} + H_\sigma^2)^{-\frac{1}{4}}\right\|_p = \epsilon^{-\frac{1}{2}} \left\|(H_\sigma + \epsilon^{-1}\imath)^{-1}\right\|_{\frac{p}{2}}^2$$

where that final norm is also finite and monotonously decreasing to 0 with $\epsilon \to 0$ if $\frac{p}{2} > \frac{d}{2}$ (which follows e.g. from the monotone convergence theorem that also applies to the non-commutative L^p-norms, see e.g. [54]).

For the difference $\varphi_\epsilon(\hat{H}_I) - \mathcal{P}_+ \varphi_\epsilon(H_+) - \mathcal{P}_- \varphi_\epsilon(H_-)$ one can use a similar scaling argument to obtain

$$\left\|\varphi_\epsilon(\hat{H}_I) - \mathcal{P}_+ \varphi_\epsilon(H_+) - \mathcal{P}_- \varphi_\epsilon(H_-)\right\|_{\mathscr{L}_{\infty,K}} \leq C_K$$

uniformly in ϵ. To complete the proof using log-convexity it is again enough to show that the local L^p-norms converge to 0, hence we need to bound the scaling behavior of

$$\left\|P_x (\epsilon H_\sigma + \imath)^{-1} (\epsilon H_\sigma - \epsilon \hat{H}_I)(\epsilon \hat{H}_I + \imath)^{-1} P_y\right\|_p$$

$$\leq \epsilon \left\| P_x(1+\epsilon^2 H_\sigma^2)^{-\frac{1}{2}} \right\|_{2p} \left\| (1+\epsilon^2 \hat{H}_I^2)^{-\frac{1}{2}} P_y \right\|_{2p} \|\hat{H}_I - H_\sigma\|$$
$$\leq \epsilon \left\| P_x(1+\epsilon^2 H_\sigma^2)^{-\frac{1}{4}} \right\|_{2p} \left\| (1+\epsilon^2 \hat{H}_I^2)^{-\frac{1}{4}} P_y \right\|_{2p} \|\hat{H}_I - H_\sigma\|$$
$$= \epsilon \left\| P_x(1+\epsilon^2 H_\sigma^2)^{-\frac{1}{2}} P_x \right\|_p^{\frac{1}{2}} \left\| P_y(1+\epsilon^2 \hat{H}_I^2)^{-\frac{1}{2}} P_y \right\|_p^{\frac{1}{2}} \|\hat{H}_I - H_\sigma\|$$
$$\leq C \left\| P_x(\epsilon^{-2}+H_\sigma^2)^{-\frac{1}{2}} P_x \right\|_p^{\frac{1}{2}} \left\| P_y(\epsilon^{-2}+\hat{H}_I^2)^{-\frac{1}{2}} P_y \right\|_p^{\frac{1}{2}}$$

where we in the second inequality used $\|(1+\epsilon^2 H_\sigma^2)^{-1}\| \leq 1$ to get rid of half of the fractional power. The final expression converges to 0 for $\epsilon \to 0$ provided $p > \frac{d}{2}$ due to monotone convergence. □

For the infinite-volume limit we need another technical result:

Lemma 6.2.14 *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of p-smooth elements. We say that a_n converges to some p-coefficient-wise to some if $P_x a_n P_y \to P_x a P_y$ holds in p-norm for all $x, y \in \mathbb{Z}^d$.*

(i) Let there be two sequences $(a_n)_{n\in\mathbb{N}}$, $(a_n)_{n\in\mathbb{N}}$of p_1-smooth respectively p_2-smooth elements which converge p_i-coefficient-wise to a respectively b. If all seminorms $\|a_n\|_{\mathscr{S}_{p_i,m}}$, $\|b_n\|_{\mathscr{S}_{p_i,m}}$ are bounded independently of n then $a_n b_n \to ab$ converges p-coefficient wise for $p^{-1} = p_1^{-1} + p_2^{-1}$.

(ii) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of p-localized elements converging p-coefficient-wise to some a and all seminorms $\|a_n\|_{\mathscr{L}_{p,k}}$ are bounded uniformly in n. Then $a_n \to a$ converges in the Fréchet topology of $\mathscr{L}_p$, in particular in L^p-norm.

Proof. To (i) write out $P_x a_n b_n P_y = \sum_{z\in\mathbb{Z}^d} P_x a_n P_z b_n P_y$ which converges absolutely in p-norm and apply dominated convergence. Similarly in (ii) write $a = \sum_{x,y\in\mathbb{Z}^d} P_x a_n P_y$ as an absolutely convergent sum. □

6.2.3 Proofs for the mobility-gapped case

Proof (of Lemma 6.2.3). The idea is to shift with the dual action which replaces $\hat{\xi}_{\sigma L}(\overline{\Pi}_{L\sigma}) = \overline{\Pi}_{0\sigma}$ but doesn't change the trace. Thus one needs to prove that in the norm of $L^1(\mathcal{E})$ we have the convergence

$$\lim_{m\to\sigma\infty} \hat{\xi}_m \left(\varphi(\hat{H}_I)(\overline{\Pi}^{(t)}_{L\sigma} - \overline{\Pi}_{L\sigma})[G(\hat{H}_I), X_w]\varphi(\hat{H}_I)\right)$$
$$= \varphi(H_\sigma)(\overline{\Pi}^{(\sigma,t)}_{0\sigma} - \overline{\Pi}_{0\sigma})[G(H_\sigma), X_w]\varphi(H_\sigma)$$

and similarly for the integral term. Due to Lemma 6.2.13 one of the factors is ∞-localized uniformly in L for any $p \in [1,\infty]$, namely

$$\hat{\xi}_m \left((\overline{\Pi}^{(t)}_{L\sigma} - \overline{\Pi}_{L\sigma})\right) = \hat{\xi}_m(e^{\imath F(\hat{H}_I)t}[\overline{\Pi}_{0\sigma}, \hat{\xi}_m(e^{-\imath F(\hat{H}_I)t})]$$

Since all factors are uniformly p-smooth for some p, Lemma 6.2.14 implies that it is enough to prove that each factor converges p-coefficient-wise for some values of p reciprocals add up to 1. Indeed, that is not difficult to see for the various functions of $\hat{H}_I$ due to the resolvent-identity, since

$$P_x\hat{\xi}_m\left(\frac{1}{\hat{H}_I+\imath}\right)P_y = P_x\left(\frac{1}{H_\sigma+\imath}\right)P_y + P_x\hat{\xi}_m\left(\frac{1}{\hat{H}_I+\imath}\right)(\hat{\xi}_m(\hat{V}) - V_\sigma)\frac{1}{\hat{H}_I+\imath}P_y$$

and the perturbation $\hat{\xi}_m(\hat{V}) - V_\sigma$ is essentially only supported at a halfspace $X \cdot v < m$, far away from x, y for large m. In the end, L^1-convergence therefore follows by dominated convergence, as argued in Lemma 6.2.14.

For the integral term it is by another dominated convergence argument enough to prove that the integrand converges pointwise in L^1-norm. Here the uniform ∞-localization is provided by the commutator

$$\hat{\xi}_m\left([\overline{\Pi}^{(t)}_{L\sigma}, \hat{H}_I]\frac{1}{\hat{H}_I - z}\right)$$

where we recall that its ∞-localization is one of our assumptions. Again all factors are uniformly p-smooth and converge coefficientwise. For that it is helpful to note that

$$\hat{\xi}_m([\hat{H}_I, X_w]) - [H_\sigma, X_w]) = [\hat{\xi}_m(\hat{V}) - V_\sigma, X_w]$$

is a bounded operator since $\hat{\xi}$ leaves the unbounded part invariant.

Finally, we need to check that the error term Z_L converges individually to the stated Z_∞. Key to this is that we can write

$$\hat{\mathcal{T}}_\xi(Z_L(\varphi, w, t, g; \hat{H}_I)) = \hat{\mathcal{T}}_\xi(\varphi(\hat{H}_I)\Pi_L[G(\hat{H}_I), X_w]\varphi(\hat{H}_I)) \\ - \sum_\sigma \hat{\mathcal{T}}_\xi(\varphi(H_\sigma)\Pi_L P_\sigma[G(H_\sigma), X_w]\varphi(H_\sigma)).$$

Here we used that the second line in the definition (6.2.4) drops out since its trace vanishes identically and instead added another term with vanishing trace (as will be proven in the following lemma). Convergence of this term in $L^1(\mathcal{E})$ follows from Lemma 1.3.1 since the limit $L \to \infty$ is the SOT-limit of $\Pi_L \to \mathbb{1}$ and

$$\varphi(\hat{H}_I)[G(\hat{H}_I), X_w]\varphi(\hat{H}_I) - \sum_\sigma \varphi(H_\sigma)P_\sigma[G(H_\sigma), X_w]\varphi(H_\sigma)$$

is 1-localized due to Lemma 6.2.13. □

The technical point left out above is a variation of an argument that is well-known in the case for Hamiltonians that are bounded from below (see e.g. [16, 76]):

Lemma 6.2.15 *Assume that H_σ has a mobility gap in Δ with $\mathrm{supp}(g) \subset \Delta$ or that H_σ is bounded from below.*

For any L^1-function f and cutoff φ with $\varphi g = g$ one has

$$\hat{\mathcal{T}}_\xi(\varphi(H_\sigma) f(X_v)[G(H_\sigma), X_w]\varphi(H_\sigma)) = 0.$$

Proof. Due to Corollary 2.1.4 one has

$$\hat{\mathcal{T}}_\xi(\varphi(H_\sigma) f(X_v)[G(H_\sigma), X_w]\varphi(H_\sigma)) = \|f\|_{L^1(\mathbb{R})}\, \mathcal{T}(\varphi(H_\sigma)[G(H_\sigma), X_w]\varphi(H_\sigma)).$$

In the mobility gaped case one can write $G(H_\sigma) = \int_\mathbb{R} g(\lambda)e_{\sigma\lambda}\mathrm{d}\lambda$ using the spectral decomposition $e_{\sigma\lambda} = \chi(H_\sigma < \lambda)$ and $e_{\sigma\lambda}$ depends continuously on λ w.r.t. to any $\mathcal{S}_{p,n}$ with $p \in (\frac{d}{2}, \infty)$ due to the mobility gap. More precisely, after subtracting the constant part $e_{\sigma\lambda} - e_- \in W^\infty_p(\mathcal{A})$ is in fact continuous w.r.t. each of the seminorms $\mathscr{S}_{p,K}$ for all $1 \le p < \infty$ since $\|e_{\sigma\lambda} - e_{\sigma\lambda'}\|_1 \le C\,|\lambda - \lambda'|^\gamma$ due to the assumed Hölder-continuity of the DOS-measure). Therefore

$$\mathcal{T}(\varphi(H_\sigma)[G(H_\sigma), X_w]\varphi(H_\sigma)) = \int_\mathbb{R} g(\lambda)\mathcal{T}(\varphi(H_\sigma)[e_{\sigma\lambda}, X_w]\varphi(H_\sigma))\mathrm{d}\lambda$$

and the right-hand side vanishes by cyclicity since

$$[e_{\sigma\lambda}, X_w] = e_{\sigma\lambda}[e_{\sigma\lambda}, X_w]e_{\sigma\lambda}^{\perp} + e_{\sigma\lambda}^{\perp}[e_{\sigma\lambda}, X_w]e_{\sigma\lambda}.$$

In the case that H_σ is bounded from below we instead note that due to (6.2.1) and(6.2.2) one can write

$$\begin{aligned}\mathcal{T}(\varphi(H_\sigma)[G(H_\sigma), X_w]\varphi(H_\sigma)) &= \mathcal{T}(\varphi(H_\sigma)[H_\sigma, X_w]g(H_\sigma)\varphi(H_\sigma)) \\ &= \mathcal{T}([H_\sigma, X_w]g(H_\sigma))\end{aligned}$$

with the final equality holding since $\varphi g = g$. As $G(H_\sigma)$ is trace-class one can also on the left-hand side take the SOT-limit $\varphi(H_\sigma) \to \mathbb{1}$ which gives

$$\mathcal{T}(\varphi(H_\sigma)[G(H_\sigma), X_w]\varphi(H_\sigma)) = \mathcal{T}([G(H_\sigma), X_w]) = 0$$

since $G(H_\sigma) \in M_N(W_1^1(\mathcal{A}))^\sim$ and the $\mathcal{T}(\nabla_w a) = 0$ due to θ-invariance of $\mathcal{T}$.

□

The next step is to prove that the error term in Lemma 6.2.3 vanishes.

Proof (of Lemma 6.2.4). If $\hat{H}_I$ is bounded from below then this is comparatively simple, for then

$$\begin{aligned}&\lim_{\varphi\to\mathbb{1}} \hat{\mathcal{T}}_\xi(Z_\infty(\varphi, w, t, g; \hat{H}_I)) \\ &\quad= \sum_{\sigma\in\{-,+\}} \hat{\mathcal{T}}_\xi(\overline{\Pi}_{0\sigma}[G(\hat{H}_I), X_w]) - \overline{\Pi}_{0\sigma}[G(H_\sigma), X_w]) \\ &\quad= \sum_{\sigma\in\{-,+\}} \sum_{m\in\mathbb{Z}} \hat{\mathcal{T}}_\xi(P_m\overline{\Pi}_{0\sigma}[G(\hat{H}_I), X_w] - P_m\overline{\Pi}_{0\sigma}[G(H_\sigma), X_w]) \\ &\quad= \sum_{\sigma\in\{-,+\}} \sum_{m\in\mathbb{Z}} \hat{\mathcal{T}}_\xi(P_m\overline{\Pi}_{0\sigma}[G(\hat{H}_I), X_w]) - \hat{\mathcal{T}}_\xi(P_m\overline{\Pi}_{0\sigma}[G(H_\sigma), X_w]) \\ &\quad= \sum_{\sigma\in\{-,+\}} \sum_{m\in\mathbb{Z}} \hat{\mathcal{T}}_\xi([P_m\overline{\Pi}_{0\sigma}G(\hat{H}_I), X_w]) - \hat{\mathcal{T}}_\xi([P_m\overline{\Pi}_{0\sigma}G(H_\sigma), X_w])\end{aligned}$$

where we used that (since G restricts to a Schwartz function) the difference is trace-class in $L^1(\mathcal{E})$ and $\varphi(H_\sigma)$ converges to $\mathbb{1}$ strongly, then we inserted a partition of unity and split the trace because all summands are separately trace-class. Finally, each of those traces vanishes identically since to the invariance of $\hat{\mathcal{T}}_\xi$ implies $\hat{\mathcal{T}}_\xi(\nabla_w a) = 0$ for any $a \in W_1^1(\mathcal{E})$.

In the other case where $d = 2$ and $\hat{H}_I$ is not necessarily bounded from below we need to make a careful scaling argument since $[G(H_\sigma), X_w]$ is not always L^1 (morally speaking, the limit should still vanish since it is formally the trace of a total derivative).

As a first step we note that the L^1-limit of

$$(\varphi_\epsilon(\hat{H}_I) - \varphi_\epsilon(H_\sigma))\overline{\Pi}_{0\sigma}[G(H_\sigma), X_w]$$

vanishes. That follows from Lemma 6.2.13(iii,v) since $[G(H_\sigma), X_w]$ is (2+)-smooth and $\varphi(\epsilon\hat{H}_I) - \varphi(\epsilon H_\sigma)$ is (2−)-localized with seminorms that vanish as $\epsilon \to 0$.

Hence the limit of interest is the same as

$$\lim_{\epsilon\to 0} \sum_{\sigma\in\{-,+\}} \hat{\mathcal{T}}_\xi(\varphi_\epsilon(\hat{H}_I)\left(\overline{\Pi}_{0\sigma}[G(\hat{H}_I), X_w] - \overline{\Pi}_{0\sigma}[G(H_\sigma), X_w]\right)\varphi_\epsilon(\hat{H}_I)).$$

Applying partial integration it is sufficient to prove that

$$\lim_{\epsilon\to 0}\left\|(\nabla_w\varphi_\epsilon(\hat{H}_I))\left(G(\hat{H}_I) - \sum_{\sigma\in\{-,+\}}\overline{\Pi}_{0\sigma}G(H_\sigma)\right)\right\|_1 = 0$$

and indeed that also follows from Lemma 6.2.13(v) since $\nabla_w\varphi_\epsilon(\hat{H}_I)$ is (2+)-smooth with seminorms converging to 0 and the factor involving G is (2−)-localized.

□

The scaling argument does not work out for $d = 3$, hence we need to impose the existence of a lower bound on H_σ there.

Proof (of Lemma 6.2.5). There is the general identity

$$\hat{\mathcal{T}}_\xi(a[\Pi_{0\sigma}, b]c) = -\sigma\hat{\mathcal{T}}_\xi(a[X_w, b]c)$$

valid for bulk elements a, b, c for a variety of regularity conditions, for example 1-smooth a, b in $L^\infty(\mathcal{A})$ and ∞-smooth $b \in L^\infty(\mathcal{A})$. We note that it is enough to prove the equality under the assumption that $\Pi_{0\sigma}$ is an exact stepfunction, since one can write an arbitrary $\Pi_{0\sigma}$ as an integral of such stepfunctions. But for a stepfunction one can read off the identity as a special case of Proposition 2.3.4 for the equality of the two 1-cocycles

$$-\frac{1}{2}\hat{\mathcal{T}}_{\xi}(a[\operatorname{sgn}(X_v), b]) = \mathcal{T}(a[X_v, b])$$

since X_v here is the generator of the crossed product; the same as the Dirac-operator for the case of a one-dimensional action in section 2). □

Proof (of Lemma 6.2.6). The integral converges absolutely in $L^1(\mathcal{A})$ and by dominated convergence the time average can be taken pointwise. After bracketing one of the resolvents one has the integrand

$$\varphi(H_\sigma)\frac{1}{H_\sigma - z}\langle [X_w, H_\sigma]\frac{1}{H_\sigma + \iota}\rangle_T\,(1 + \frac{z-\iota}{H_\sigma - z})[H_\sigma, X_w]\frac{1}{H_\sigma - z}\varphi(H_\sigma)$$

and, by the L^p-ergodic theorem (see Corollary 1.3.3) the time-average converges in L^p-norm for any $3 < p < \infty$ and then takes the form

$$\varphi(H_\sigma)\frac{1}{H_\sigma - z}A_\sigma\,(1 + \frac{z-\iota}{H_\sigma - z})[H_\sigma, X_w]\frac{1}{H_\sigma - z}\varphi(H_\sigma)$$

with A_σ some operator $A_\sigma \in L^p(\mathcal{A})$ that commutes with H_σ. We decompose $1 = e_{\sigma-} + e_{\sigma+} + e_{\sigma\Delta}$. Upon projecting with $e_{\sigma\pm}$ the integrand becomes

$$(\partial_{\overline{z}}\tilde{G}_K)(z)\varphi(H_\sigma)\frac{e_{\sigma\pm}}{e_{\sigma\pm}H_\sigma - z}A_\sigma\,(1 + \frac{z-\iota}{e_{\sigma\pm}H_\sigma - z})[H_\sigma, X_w]\frac{1}{e_{\sigma\pm}H_\sigma - z}\varphi(H_\sigma)$$

and we recall that on the real line itself $(\partial_{\overline{z}}\tilde{G}_K)(z)$ is by construction only supported on a subset of Δ. Thus the resolvents are analytic everywhere on the support and (6.2.3) is actually a total $\partial_{\overline{z}}$-derivative. Performing the $\overline{z}$-integral first the integral thus vanishes identically.

It remains the $e_{\sigma\Delta}$-term

$$\varphi(H_\sigma)\frac{1}{H_\sigma - z}e_{\sigma\Delta}A_\sigma e_{\sigma\Delta}\,(1 + \frac{z-\iota}{H_\sigma - z})[H_\sigma, X_w]\frac{1}{H_\sigma - z}\varphi(H_\sigma)$$

which we now show also vanishes due to the mobility gap hypothesis. We note that

$$e_{\sigma\Delta}e^{\iota F(H_\sigma)t}[H_\sigma, X_v]e_{\sigma\Delta}e^{-\iota F(H_\sigma)t} = e_{\sigma\Delta}e^{\iota F(H_\sigma)t}[\phi(H_\sigma), X_d]e_{\sigma\Delta}e^{-\iota F(H_\sigma)t}$$

for any compactly supported smooth function ϕ with $H_\sigma e_{\sigma\Delta} = \phi(H_\sigma)e_{\sigma\Delta}$. Furthermore we can expand

$$[\phi(H_\sigma), X_v] = \int_{\mathbb{C}} (\partial_{\bar{z}}\tilde{\phi}_K)(z)(F(H_\sigma) - z)^{-1}[F(H_\sigma), X_v](F(H_\sigma) - z)^{-1}\mathrm{d}z \wedge \mathrm{d}\bar{z}$$

since $\phi \circ F^{-1}$ is also a rapidly decaying function.

We also have

$$\int_0^T e^{\imath F(H_\sigma)t}[F(H_\sigma), X_d]e^{-\imath F(H_\sigma)t}\mathrm{d}t = e^{\imath F(H_\sigma)T}[X_d, e^{-\imath F(H_\sigma)T}]$$

which is as usual checked by differentiation of the right-hand side. This is further manipulated to

$$e_\Delta e^{\imath F(H_\sigma)T}[X_d, e^{-\imath F(H_\sigma)T}]e_\Delta = e_\Delta e^{\imath F(H_\sigma)T}[X_d, e^{-\imath F(H_\sigma)T}e_\Delta]e_\Delta - e_\Delta[X_d, e_\Delta]e_\Delta$$

and then the uniform boundedness of derivatives $[e^{-\imath F(H_\sigma)T}e_\Delta, X_d]$ in L^p-norm due to the mobility gap hypothesis means that

$$\begin{aligned}\|e_\Delta A_\sigma e_\Delta\|_1 &\le \lim_{T\to\infty}\frac{1}{T}\left\|\int_0^T e_\Delta e^{\imath F(H_\sigma)t}[H_\sigma, X_d]\frac{1}{H_\sigma + \imath}e^{-\imath F(H_\sigma t)}e_\Delta\mathrm{d}t\right\|_1 \\ &\le C\frac{1}{T}\left\|\int_0^T e_\Delta e^{\imath F(H_\sigma)t}[F(H_\sigma), X_d]e^{-\imath F(H_\sigma)t}e_\Delta\mathrm{d}t\right\|_1 = \lim_{T\to\infty}\frac{1}{T}c.\end{aligned}$$

□

Lemma 6.2.16 *Let e, e_* be spectral projections of H_σ which lie in $M_N(W_2^1(\mathcal{A}))^\sim$ with either $ee_* = e_*$ or $e^\perp e_* = e_*$ then*

$$\begin{aligned}&\lim_{T\to\infty}\mathcal{T}(\varphi(H_\sigma)e_*e^{-\imath F(H_\sigma)t}[X_d, e^{-\imath F(H_\sigma)t}][e, X_w]e_*\varphi(H_\sigma)) \\ &= \mathcal{T}(\varphi(H_\sigma)e_*[X_d, e_*][e, X_w]e_*\varphi(H_\sigma))\end{aligned}$$

where we interpret $[e, X_w] \sim \nabla_w e$ as the derivative in $L^2(\mathcal{A})$-norm.

Proof. Assume $ee_* = e_*$ as the other case is identical (note that e only appears inside a commutator). Using the off-diagonality of the commutator

$$e_*e^{\imath F(H_\sigma)t}[X_d, e^{-\imath F(H_\sigma)t}][e, X_w]e_* = e_*ee^{-\imath F(H_\sigma)t}[X_d, e^{-\imath F(H_\sigma)t}](1 - e)[e, X_w]e_*$$

$$
\begin{aligned}
&= e_* e^{\imath F(H_\sigma)t}[[e, X_d], e^{-\imath F(H_\sigma)t}](1-e)[e, X_w]e_* \\
&= e_*[e, X_d][e, X_w]e_* - e_* e^{\imath F(H_\sigma)t}[e, X_d]e^{-\imath F(H_\sigma)t}[e, X_w]e_* \\
&= e_*[e_*, X_d][e, X_w]e_* - e_* e^{\imath F(H_\sigma)t}[e_*, X_d]e^{-\imath F(H_\sigma)t}[e, X_w]e_*
\end{aligned}
$$

where the last line is a consequence of

$$
\begin{aligned}
e_*[e-e_*, X_d][e, X_w]e_* = {} & e_*(e-e_*)[e-e_*, X_d](e-e_*)^\perp e^\perp[e, X_w]e_* \\
& + e_*(e-e_*)^\perp[e-e_*, X_d](e-e_*)e^\perp[e, X_w]e_*
\end{aligned}
$$

each term of which is zero. The time average exists in L^2-sense and eliminates the t-dependent term, again due to off-diagonality since all spectral projections of H_σ commute with the limit. □

Lemma 6.2.17 *Let $e \in M_N(W_2^1(\mathcal{A})^\sim$ be a projection and let $\mathbb{1} = \bigoplus_{k=1}^m e_k$ with projections $e_k \in M_N(W_2^1(\mathcal{A})^\sim$ that commute with e and such that for each k either $ee_k = e_k$ or $e^\perp e_k = e_k$ holds. Then*

$$
\frac{1}{2\pi\imath} Ch_{v\times w}(e) = \sum_{k=1}^m \mathcal{T}(e_k[e, X_w][e_k, X_v]). \tag{6.2.10}
$$

Proof. One starts from

$$
\begin{aligned}
\frac{1}{2\pi\imath}\mathrm{Ch}_{v\times w}(e) &= \mathcal{T}((e - s(e))[[e, X_w][e, X_v]]) \\
&= \mathcal{T}(e[[e, X_w][e, X_v]]) \\
&= \mathcal{T}(e[e, X_w][e, X_v]) - \mathcal{T}([e, X_v]e^\perp[e, X_w]) \\
&= \mathcal{T}(e[e, X_w][e, X_v]) - \mathcal{T}(e^\perp[e, X_w][e, X_v]e^\perp) \\
&= \sum_{k=1}^m \mathcal{T}(e_k e[e, X_w][e, X_v]e_k) - \mathcal{T}(e_k e^\perp[e, X_w][e, X_v]e^\perp e_k)
\end{aligned}
$$

where we used that the scalar part of e drops out due to the algebraic properties of cyclic cocycles and that the derivative of a projection is off-diagonal $[e, X_w] = e^\perp[e, X_w]e + e[e, X_w]e^\perp$ due to the Leibniz identity.

By assumption for each k one of the two terms vanishes and in the case $ee_k = e_k$, equivalently $e^\perp e_k = 0$, one has also

$$
0 = [e^\perp e_k, X_w] = e^\perp[e_k, X_w] + [e^\perp, X_w]e_k = e^\perp[e_k, X_w] - [e, X_w]e_k
$$

such that

$$\begin{aligned}\mathcal{T}(e_k e[e,X_w][e,X_v]e_k) &= \mathcal{T}(e_k e[e,X_w]e^{\perp}[e_k,X_v]e_k)\\ &= \mathcal{T}(e_k e[e,X_w][e_k,X_v]e_k) = \mathcal{T}(e_k[e,X_w][e_k,X_v]e_k).\end{aligned}$$

In the other case $e^{\perp}e_k = e_k$ one instead has

$$0 = [ee_k, X_w] = e[e_k, X_w] + [e, X_w]e_k$$

which leads to

$$-\mathcal{T}(e_k e^{\perp}[e,X_w][e,X_v]e_k) = \mathcal{T}(e_k[e,X_w][e_k,X_v]e_k).$$

□

Proof (of Lemma 6.2.7) The time average of the integral term vanishes by Lemma 6.2.6. Into

$$\mathcal{T}\left(\varphi(H_\sigma)e^{\iota F(H_\sigma)t}[X_d, e^{-\iota F(H_\sigma)t}]\,[G(H_\sigma), X_w]\varphi(H_\sigma)\right)$$

one inserts the identity $G(H_\sigma) = \int_\Delta g(\lambda)e_{\sigma\lambda}\mathrm{d}\lambda$ which converges absolutely in the norm of $M_N(W_2^1(\mathcal{A}))^{\sim}$ as argued before.

Thus the time-average limit can be computed pointwise in λ, which we do by inserting the decomposition $1 = \sum_{k=1}^4 e_k$ with

$$e_1 = e_-,\ e_2 = e_+,\ e_3 = e_{\sigma\Delta}e_{\sigma\lambda},\ e_4 = e_{\sigma\Delta}e_{\sigma\lambda}^{\perp}$$

and invoking Lemma 6.2.16 four times to get

$$\begin{aligned}&\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathcal{T}(\varphi(H_\sigma)e^{-\iota F(H_\sigma)t}[X_d, e^{-\iota F(H_\sigma)t}][e_{\sigma\lambda}, X_w]\varphi(H_\sigma))\\ &= \sum_{k=1}^4 \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathcal{T}(\varphi(H_\sigma)e_k e^{-\iota F(H_\sigma)t}[X_d, e^{-\iota F(H_\sigma)t}][e_{\sigma\lambda}, X_w]e_k\varphi(H_\sigma))\\ &= \sum_{k=1}^4 \mathcal{T}(\varphi(H_\sigma)e_k[X_d, e_k][e_{\sigma\lambda}, X_w]e_k\varphi(H_\sigma))\end{aligned}$$

Taking the SOT-limit $\varphi(H_\sigma) \to \mathbb{1}$ for $\varphi \to \mathbb{1}$ is trivial and hence the formula from Lemma 6.2.17 gives

$$\lim_{\varphi\to\mathbb{1}} \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathcal{T}(\varphi(H_\sigma)e^{-\imath F(H_\sigma)t}[X_d, e^{-\imath F(H_\sigma)t}][e_{\sigma\lambda}, X_w]\varphi(H_\sigma))$$
$$= -\sum_{k=1}^{4} \mathcal{T}(\varphi(H_\sigma)e_k[e_k, X_d][e_{\sigma\lambda}, X_w]e_k) = -\frac{1}{2\pi\imath}\mathrm{Ch}(e_{\sigma\lambda}).$$

□

6.2.4 Proofs for the pseudogapped case

We skip a few details in the pseudogapped case to avoid duplication and since all essential ingredients were seen before.

Proof. (of Lemma 6.2.9) We introduced the cutoff $\varphi(\epsilon\hat{H}_\sigma)$ as before but since g_β is not compactly supported we must immediately take the limit of $\epsilon \to 0$ to remove it. Indeed, the mollification is only necessary to justify the cyclic permutations under the trace to derive the expression

$$\hat{\mathcal{T}}_\xi(\Sigma_L(\varphi, X_w, t, g_\beta; \hat{H}_I)) = \sum_{\sigma\in\{-,+\}} (\overline{\Pi}^{(t)}_{L\sigma} - \overline{\Pi}_{L\sigma})[G_\beta(\hat{H}_I), X_w] \tag{6.2.11}$$
$$-\frac{1}{2\pi}\int_{D_\beta} \frac{1}{\hat{H}_I - z}[\overline{\Pi}^{(t)}_{L\sigma}, \hat{H}_I]\frac{1}{\hat{H}_I - z}[\hat{H}_I, X_w]\frac{1}{\hat{H}_I - z}\mathrm{d}\nu_\beta(z). \tag{6.2.12}$$

which is well-defined even without the cutoff (since there are enough resolvents to make the integral absolutely convergent in $L^1(\mathcal{E})$-norm and the term $(\overline{\Pi}^{(t)}_{L\sigma} - \overline{\Pi}_{L\sigma})[G_\beta(\hat{H}_I), X_w]$ is also $L^1(\mathcal{E})$ since it is polynomially localized to the interface region and $[G_\beta(\hat{H}_I), X_w]$ is locally L^1 since G_β decays exponentially on $\sigma(\hat{H})_I$. The $L \to \infty$ limit can be justified as before using dominated convergence (where it helps that the contour integral converges absolutely due to the exponential decay of G_β). □

Next we need to assert that G_β converges to the Fermi projection in an appropriate sense, which is a contour integration argument similar to Proposition 4.3.18.

Lemma 6.2.18 *Assume that H_σ has a sublinear pseudogap, i.e. it satisfies Definition 4.3.7 with $\gamma > 2$ then one has $\lim_{\beta\to\infty} \left\|e_{\sigma F} - G_\beta(H_\sigma)\right\|_{W^1_p(\mathcal{A})} = 0$ for all $p \in [1, \gamma)$.*

Proof. First off, we note that the integral $G_\beta(H_\sigma) = \frac{1}{2\pi}\int_{\partial R_\beta} G_\beta(z)\frac{1}{H_\sigma - z}\mathrm{d}z$ also converges in L^p-norm for $p \in (\frac{3}{2}, \gamma)$. Taking the limit $\beta \to \infty$ one sees that G_β restricted to ∂R_β converges to $\chi(\cdot < E_F)$ and with the exception of a small neighborhood of radius β^{-1} around the tips of the wedges convergence is uniform in z and β. Since the norm of the resolvent stays finite there due to the pseudogap estimate Proposition 4.3.17, the contributions of those arcs vanish in the limit and one has

$$\lim_{\beta\to\infty}\left\|\int_{\partial R_\beta} G_\beta(z)\frac{1}{H_\sigma - z}\mathrm{d}z - \int_{C_0}\frac{1}{H_\sigma - z}\mathrm{d}z\right\| = 0$$

where C_0 is the contour from Lemma A.6, hence the r.h.s. is indeed the Fermi projection. For the derivatives one concludes the same via $[G_\beta(H_\sigma), X_w] = -\frac{1}{2\pi}\int_{\partial R_\beta} G_\beta(z)\frac{1}{H_\sigma - z}[H_\sigma, X_w]\frac{1}{H_\sigma - z}\mathrm{d}z$, since

$$\left\|\frac{1}{H_\sigma - z}[H_\sigma, X_w]\frac{1}{H_\sigma - z}\right\|_p \leq C\,|\Im m\, z|^{-s}$$

on the wedge for some fractional $0 \leq s < 1$ depending on $p \in (\frac{3}{2}, \gamma)$ and that sublinear scaling is all that is needed for the error terms to be suppressed by the curve length in the limit.

This only renders an estimate for $p > \frac{3}{2}$ and to go to lower exponents one writes

$$G_{2\beta} = G_\beta \tilde{G}_\beta, \qquad \tilde{G}_\beta = G_{2\beta}(G_\beta)^{-1}$$

and notes that $\tilde{G}_\beta(\lambda)$ converges point-wise to $\chi(\lambda < E_F) + 2\chi(\lambda = E_F)$, i.e. $\tilde{G}_\beta(H_\sigma)$ also converges to the Fermi projection in SOT since E_F is not an eigenvalue of

H_σ. The same contour integration argument as above also passes for $\tilde{G}_\beta$, i.e. convergence is in Sobolev norm for the same range of exponents. Hence

$$\begin{aligned}\left\|e_{\sigma F} - G_{2\beta}(H_\sigma)\right\|_{W^1_q(\mathcal{A})} &= \left\|e^2_{\sigma F} - G_\beta(H_\sigma)\tilde{G}_\beta(H_\sigma)\right\|_{W^1_q(\mathcal{A})} \\ &\leq \left\|e_{\sigma F} - G_\beta(H_\sigma)\right\|_{W^1_{2q}(\mathcal{A})} \left\|\tilde{G}_\beta(H_\sigma)\right\|_{W^1_{2q}(\mathcal{A})} \\ &+ \left\|e_{\sigma F} - \tilde{G}_\beta(H_\sigma)\right\|_{W^1_{2q}(\mathcal{A})} \left\|e_{\sigma F}\right\|_{W^1_{2q}(\mathcal{A})}\end{aligned}$$

extends the range of convergence all the way to $p \in [1, \gamma)$. □

Proof (of Lemma 6.2.10). Due to Lemma 6.2.18 the $\beta \to \infty$-limit replaces G_β with the Fermi-projection

$$\begin{aligned}&\lim_{\beta\to\infty} \mathcal{T}\left(e^{\imath F(H_\sigma)t}[X_d, e^{-\imath F(H_\sigma)t}]\,[G_\beta(H_\sigma), X_w]\right) \\ &= \mathcal{T}\left(e^{\imath F(H_\sigma)t}[X_d, e^{-\imath F(H_\sigma)t}]\,[e_{\sigma F}, X_w]\right).\end{aligned}$$

Inserting $\mathbb{1} = e_F + e_F^\perp$ two applications of Lemma 6.2.16 then give

$$\begin{aligned}&\lim_{T\to\infty}\lim_{\beta\to\infty}\frac{1}{T}\int_0^T \mathcal{T}\left(e^{\imath F(H_\sigma)t}[X_d, e^{-\imath F(H_\sigma)t}]\,[e_{\sigma F}, X_w]\right) \\ &= \mathcal{T}\left(e_{\sigma F}[X_d, e_{\sigma F}][e_{\sigma F}, X_w]e_{\sigma F}\right) \\ &+ \mathcal{T}\left(e^\perp_{\sigma F}[X_d, e^\perp_{\sigma F}][e_{\sigma F}, X_w]e^\perp_{\sigma F}\right) = \frac{1}{2\pi\imath}\mathrm{Ch}_{v\times w}(e_F)\end{aligned}$$

which is the Chern number due to Lemma 6.2.17. □

Proof (of Lemma 6.2.11). We can bound

$$\left\|\frac{1}{H_\sigma - z}e^{\imath F(H_\sigma)t}[X_d, H_\sigma]e^{-\imath F(H_\sigma)t}\frac{1}{H_\sigma - z}[H_\sigma, X_w]\,\frac{1}{H_\sigma - z}\right\|_1 \leq C\,|\Im m z|^{-s}$$

for some $0 < s < 1$ on any cone including $\cap_{\beta>1} D_\beta$. To see this one notes that we have one factor of the resolvent and two of $[H_\sigma, X_v](H_\sigma - z)^{-1}$ (the former being in $L^p(\mathcal{A})$ for at least $p \in (\frac{3}{2}, \infty)$ and each of the latter for $p \in (3, \infty)$, hence the product is in $L^1(\mathcal{A})$ in the first place. The divergence close to the tip of the cone comes only from the resolvent and is of at most fractional order due to

Proposition 4.3.17 as the pseudogap of order $\gamma > 2$ allows to bound expressions such as

$$\left\| \frac{1}{H_\sigma - z} B_1 \frac{1}{H_\sigma - z} B_2 \frac{1}{H_\sigma - z} \right\|_1 \leq \|B_1\| \, \|B_2\| \, \left| \Im m\, z \right|^{-3+\gamma-\epsilon} \left\| \frac{1}{H_\sigma - z} \right\|_{\gamma-\epsilon}^{\gamma-\epsilon}$$

for small $\epsilon > 0$ and bounded operators B_1,B_2 (here we used log-convexity of the L^p-norms to interpolate the 3-norm between $\gamma - \epsilon$ and ∞).

The limit for $\beta \to \infty$ of the term under the trace thus exists in L^1-norm due to dominated convergence and is equal to

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \int_{C_0} \mathcal{T}\left(\frac{e^{\imath F(H_\sigma)t}}{H_\sigma - z} [X_d, H_\sigma] \frac{e^{-\imath F(H_\sigma)t}}{H_\sigma - z} [H_\sigma, X_w] \frac{1}{H_\sigma - z} \right) \mathrm{d}\nu_\beta(z) \mathrm{d}t$$

for C_0 again the contour from Lemma A.6. By dominated convergence the $T \to \infty$ limit can be taken pointwise and hence

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T} \int_0^T \int_{C_0} \mathcal{T}\left(\frac{e^{\imath F(H_\sigma)t}}{H_\sigma - z} [X_d, H_\sigma] \frac{e^{-\imath F(H_\sigma)t}}{H_\sigma - z} [H_\sigma, X_w] \frac{1}{H_\sigma - z} \right) \mathrm{d}z \mathrm{d}t \\
&= \int_{C_0} \mathcal{T}\left(\frac{1}{H_\sigma - z} A_\sigma \left(1 + \frac{\imath + z}{H_\sigma - z}\right) [H_\sigma, X_w] \frac{1}{H_\sigma - z} \right) \mathrm{d}z \\
&= \int_{C_0} \mathcal{T}\left(A_\sigma [H_\sigma, X_w] \left(\frac{\imath}{(H_\sigma - z)^3} + \frac{z}{(H_\sigma - z)^3} + \frac{1}{(H_\sigma - z)^2} \right) \right) \mathrm{d}z.
\end{aligned}$$

for the operator $A_\sigma = \lim_{T\to\infty} \frac{1}{T} \int_0^T e^{\imath F(H_\sigma)t} [X_d, H_\sigma] \frac{1}{H_\sigma + \imath} e^{-\imath F(H_\sigma)t} \mathrm{d}t$ where the limits exists due to the ergodic theorem in the form of Corollary 1.3.3 and commutes with H_σ. To compute the final integral we replace it again with the limit of contour integrals

$$\begin{aligned}
&\int_{C_0} \mathcal{T}\left(A_\sigma [H_\sigma, X_w] \frac{\imath + z + 1}{(H_\sigma - z)^3} \right) \mathrm{d}\nu_\beta(z) \\
&= \lim_{\beta\to\infty} \int_{D_\beta} \mathcal{T}\left(A_\sigma [H_\sigma, X_w] \frac{\imath + z + 1}{(H_\sigma - z)^3} \right) \mathrm{d}\nu_\beta(z) \\
&= \lim_{\beta\to\infty} \mathcal{T}\left(A_\sigma [H_\sigma, X_w] \left(c_1 G_\beta''(H_\sigma) + c_2 G_\beta'(H_\sigma) + c_3 G_\beta''(H_\sigma) H_\sigma + c_4 G_\beta'(H_\sigma) \right) \right) \\
&\equiv \lim_{\beta\to\infty} \mathcal{T}(M_\beta)
\end{aligned}$$

some numerical constants which result from the Cauchy integral formula. Designating the term in the final brackets as M_β we can, due to $\left\|A_\sigma[H_\sigma, X_w]\right\|_\infty < \infty$, estimate the remaining terms using the density of states

$$\left\|M_\beta\right\|_{L^1(\mathcal{A})} \leq C \left\|c_1 G''_\beta + c_2 G'_\beta + c_3 G''_\beta \mathrm{id} + c_4 G'_\beta\right\|_{L^1(\sigma(H_\sigma, \nu_{H_\sigma}))}.$$

Those exponentially decaying bump functions converge to zero outside of any compact neighborhood of E_F and around E_F are dominated by the quadratic pseudogap; by scaling the L^1-norm behaves like $O(\beta^{1-(\gamma-1)} + \beta^{-(\gamma-1)})$ with γ the order of the pseudogap. Hence M_β converges to 0 in $L^1(\mathcal{A})$, which completes the proof. □

6.2.5 Open ends

In this section we sketch briefly possible extensions and open problems related to this regularized bulk-boundary correspondence.

The first open problem is the extension to Dirac-type first order differential operators. In the spectrally gapped case we can still argue that the interface current satisfies

$$\lim_{L\to\infty} \sigma(L, X_w, 0, g; \hat{H}_I) = \hat{\mathcal{T}}_\xi([\hat{H}_I, X_w]g(\hat{H}_I)) = \langle \mathrm{Ch}_{\hat{\mathcal{T}}_\xi, w}, [\hat{u}_\Delta]_1\rangle.$$

Hence it is by the bulk-boundary correspondence Proposition 5.4.1 equal to the Chern number of the relative class of Fermi projections. We conjecture that for a Dirac-type Hamiltonian one can perform similar computations as above to obtain the result

$$\begin{aligned}
&\lim_{T\to\infty}\lim_{L\to\infty} \frac{1}{T}\int_0^T \sigma(L, X_w, t, g; \hat{H}_I)\mathrm{d}t \\
&= \int_+ \frac{1}{2\pi}\int_\Delta \mathcal{T}(e_{+\lambda}[[e_{+\lambda}, X_w], [e_{+\lambda}, X_v]] - e_{-\lambda}[[e_{-\lambda}, X_w], [e_{-\lambda}, X_v]]) \\
&\quad + \lim_{\varphi\to\mathbb{1}} \hat{\mathcal{T}}_\xi(Z_\infty(\varphi, X_w, t, g; \hat{H}_I)),
\end{aligned}$$

however, it is difficult to assert if the error term here is equal to zero (at least the scaling argument of Lemma 6.2.4 does not go through since one lacks some L^p-regularity of the resolvent). This question may be related to the problem if the formal difference Chern number on the right-hand side here is in the mobility gap regime still equal to the relative bulk index (which we defined via suspension in Section 4.3).

The second problem concerns the halfspace situation. Most of the computations above translate to Hamiltonians on a halfspace without essential modifications. Of course, we must be careful about the conditions on the halfspace model, since we know that the existence of bulk-boundary correspondence is sensitive to boundaries conditions. For a halfspace Hamiltonian $\hat{H}$ the regularized edge current is proportional to

$$\lim_{T\to\infty}\lim_{L\to\infty}\frac{1}{T}\int_0^T \hat{\mathcal{T}}_\xi\left(\{\imath[\hat{H},X_w],\Pi_L^{(t)}\}g(\hat{H})\right)\mathrm{d}t$$

where one only takes into account the contributions of a finite strip L around the boundary. The minimum that is needed for this to make sense and the argument to go through is that the resolvent of $\hat{H}$ should be locally L^p w.r.t. to the boundary trace (hence have a well-defined local density of states). In practice that is almost the same as resolvent affiliation to the halfspace algebra since the resolvents of the halfspace Hamiltonian should be equal to the restriction of the bulk resolvent up to a term that is locally L^p. If those conditions are properly formulated then one has (again for H a Hamiltonian of the quadratic type with a mobility gap in the bulk) the limit

$$\begin{aligned}\lim_{T\to\infty}\lim_{L\to\infty}\frac{1}{T}\int_0^T \sigma_L(w,t,g;\hat{H})\mathrm{d}t &= \frac{1}{2\pi}\int_\Delta g(\lambda)\mathrm{Ch}_{v\times w}(\chi(H<\lambda))\mathrm{d}\lambda \\ &\quad + \lim_{\varphi\to\mathbb{1}}\hat{\mathcal{T}}_\xi(Z_\infty(\varphi,w,t,g;\hat{H}))\end{aligned}$$

with the correction term

$$Z_\infty(\varphi,w,t,g;\hat{H}) = \varphi(\hat{H})\overline{\Pi}_{0+}[G(\hat{H}),X_w]\varphi(\hat{H}) - \varphi(H)\overline{\Pi}_{0+}[G(H),X_w]\varphi(H) \tag{6.2.13}$$

involving the bulk Hamiltonian H analogous to the one from Lemma 6.2.3. If $\hat{H}$ is bounded from below one may again argue that the correction vanishes identically. Otherwise, it should be non-trivial in some cases. From the example of quadratic

Hamiltonians with bulk gap we also know from the K-theoretic relative bulk-boundary correspondence

$$\lim_{T\to\infty}\lim_{L\to\infty}\frac{1}{2\pi T}\int_0^T \sigma_L(w,t,g;\hat{H})\mathrm{d}t = \langle \mathrm{Ch}_{\hat{\mathcal{T}}_\xi,w}, [\hat{u}_\Delta]_1\rangle$$
$$= \langle \mathrm{Ch}_{\mathcal{T},w\times v}, [e_F]_0\rangle + \langle \mathrm{Ch}_{\hat{\mathcal{T}}_\xi,w}, [U_{\mathrm{bc}}]_1\rangle$$

where $\hat{u}_\Delta$ is the edge unitary and U_{bc} is the unitary from Proposition 5.2.5 which compares the boundary conditions of $\hat{H}$ with Dirichlet boundaries. Hence the limit of (6.2.13) must be precisely the correction due to boundary conditions and may have a nontrivial topological content. It should therefore be invariant under certain transformations, e.g. under the substitution $(\hat{H},H)\to(\hat{H}+\hat{V},H+V)$ where one adds a bounded potential to the halfspace Hamiltonian and the corresponding term to the bulk Hamiltonian. In the case of quadratic Hamiltonians this could conceivably be proven by a similar argument as in Lemma 6.2.4. By reduction to the spectrally gapped case one would therefore obtain that the correction is equal to $\langle \mathrm{Ch}_{\hat{\mathcal{T}}_\xi,w}, [U_{\mathrm{bc}}]_1\rangle$ even in the mobility gapped case. Proving these relations as well as expressing the correction in terms of the boundary conditions without resorting to K-theory appears to be a formidable task.

6.3 Examples

6.3.1 Tight-binding Hamiltonians

For a model with pseudogap that has a non-trivial one-dimensional weak Chern number we can simply revisit the tight-binding model of a honeycomb lattice of Section 5.3.1. Restricting it to a halfspace in 1-direction gives with the chosen parametrization so-called zigzag boundaries and applying Theorem 6.1.8 one has a flat band of zero-energy modes with signed density $\frac{1}{3}$. A halfspace in 2-direction corresponds to armchair boundaries and the signed density of zero-energy modes is 0. Generically that will mean that there are no exact zero modes, though of course one can force them by extreme fine-tuning of the boundary potential. For more details and interpretation of this model we refer to [111, Section 5.7]

For the edge currents let us also give an example of a three-dimensional Weyl-semimetal, which has a quadratic pseudogap and non-trivial two-dimensional

Chern numbers for some directions. With the directional shifts u_i and Pauli matrices τ_i and θ a free parameter a well-known toy-model and its Fourier transform are given by[11]

$$H = iu_1\frac{\tau_1}{2} + iu_2\frac{\tau_2}{2} + (2 + e^{i\theta} + u_1 + u_2 + u_3)\frac{\tau_3}{2} + \text{h.c.}$$
$$H_k = \sin(k_1)\tau_1 + \sin(k_2)\tau_2 + (2 + \cos\theta - \cos(k_1) - \cos(k_2) - \cos(k_3))\tau_3.$$

The spectrum consists of two bands which touch at the Fermi energy $E_F = 0$ in the two so-called Weyl-points located at $(0, 0, \pm\theta)$. Around those points the dispersion relation is linear, thus one has a pseudogap of order $\gamma = 3$. The weak Chern number $\mathrm{Ch}_{v\times w}(e_F)$ for two orthogonal direction v,w is therefore well-defined and comes out to be proportional to the projected distance of the two Weyl points onto the plane spanned by v and w. When restricted to a halfspace one has so-called Fermi arc surface states that connect the Weyl points in the surface Brillouin zone. Those surface states have a large chiral velocity which is also fairly robust under disorder [131].

6.3.2 Quadratic Hamiltonians

In this section we prove that there is a class of quadratic Hamiltonians which satisfies the conditions posed in the previous sections, thereby showing that the results for unbounded Hamiltonians are not vacuously true. We also cover some technical details that were skipped over in the example sections of Chapter 4.

The most important technical tool is the convergence of domain wall Hamiltonians. For that we need a modification of resolvent convergence appropriate for the case that the resolvents converge to 0 on a closed subspace (hence the limit is not quite the resolvent of a self-adjoint operator).

Definition 6.3.1 *Let H_n be a sequence of densely defined self-adjoint operators on Hilbert space $\mathcal{H}$ and let $\hat{H}$ be densely defined and self-adjoint on $P\mathcal{H}$ with P the projection to a closed subspace.*

We say that H_n converges to $\hat{H}$ in a modified (strong) resolvent sense if

$$(H_n + z)^{-1} - (\hat{H} + z)^{-1}P \to 0 \qquad (6.3.1)$$

converges in norm (strong operator topology) for all $z \in \mathbb{C} \setminus \mathbb{R}$.

We consider $\hat{H}$ to act on all of $\mathcal{H}$ by extending with 0 on $P^\perp\mathcal{H}$. With this convention

$$(\hat{H}+z)^{-1} = P(\hat{H}+z)^{-1}P + P^\perp z^{-1}$$

and since P commutes with $(\hat{H}+z)^{-1}$ one sees that (6.3.1) is equivalent to

$$(H_n+z)^{-1} - P(\hat{H}+z)^{-1} \to 0. \tag{6.3.2}$$

The following result is standard for the usual resolvent convergence (see e.g. [106, Theorem VIII.19])

Lemma 6.3.2 *If (6.3.1) holds with norm-convergence for some $z_0 \in \mathbb{C}\setminus\mathbb{R}$ then it holds for all $z \neq \mathbb{C}\setminus\mathbb{R}$. In particular for each $\mu \neq 0$ a universal constant c_μ such that*

$$\left\|(H_n+\imath)^{-1} - P(\hat{H}+\imath)^{-1}\right\| \leq c_\mu \left\|(H_n+\imath\mu)^{-1} - P(\hat{H}+\imath\mu)^{-1}\right\|.$$

Proof. One can expand both resolvents into power series around z_0 with radius of convergence equal to $|\Im m\, z_0|$, in particular

$$P(\hat{H}+z)^{-1} = \sum_{k=0}^{\infty}(z_0-z)^k(P(\hat{H}+z)^{-1})^{k+1}$$

since P commutes with the resolvent. Then

$$\begin{aligned}&\left\|(H_n+z_0)^{-k} - \left(P(\hat{H}+z_0)^{-1}\right)^k\right\| \\ &\leq k\,|\Im m\, z_0|^{-k+1}\left\|(H_n+z_0)^{-1} - P(\hat{H}+z_0)^{-1}\right\|\end{aligned}$$

implies that (6.3.1) holds for all z in a ball of radius $|\Im m\, z_0|$ around z_0. One can iterate this procedure to obtain convergence on the half-plane containing z_0 and by conjugation (which is norm-continuous) one concludes convergence on the opposite half-plane. The constant c_μ that is mentioned can be chosen for $\mu > 1$ as

$$c_\mu = \sum_{k=1}^{\infty} k\frac{|\mu-1|^k}{|\mu|^{k-1}}$$

and for $\mu < 1$ one has to iterate the estimate. □

Proposition 6.3.3 *Let $H_n \to \hat{H}$ converge in modified resolvent sense on the range of a projection P and assume further that* $\mathrm{Dom}(H_n)$ *does not depend on n and that* $\mathrm{Dom}(\hat{H}) \subset \mathrm{Dom}(H_n)$. *Let V be a symmetric operator on* $\mathrm{Dom}(H_n)$ *and denote its restriction to* $\mathrm{Dom}(H_n)$ *by $V_P = V|_{\mathrm{Dom}(\hat{H})}$. If V satisfies the Kato-Rellich bound w.r.t. H_n (uniformly in n) respectively and V_P w.r.t. $\hat{H}$ then $H_n + V$ and $\hat{H} + V_P$ are self-adjoint.*

If $\mathrm{Dom}(\hat{H}) \subset \mathrm{Dom}(H_m)$ *then $H_n + V \to \hat{H} + V_P$ converges in modified resolvent sense.*

Proof. The convergence follows immediately from the resolvent identity

$$\begin{aligned}
&\frac{1}{H_n + V + \imath\mu} - P\frac{1}{H_n + V_P + \imath\mu} \\
&= \left(1 + \frac{1}{H_n + \imath\mu}V\right)^{-1}\left(P\frac{1}{\hat{H} + \imath\mu} - \frac{1}{H_n + \imath\mu}\right)\left(VP\frac{1}{\hat{H} + V_P + \imath\mu} - 1\right)
\end{aligned}$$

which is valid for any $\mu > 0$ so large that the $\left\|\frac{1}{H_n+V+\imath\mu}V\right\| < 1$ (and such a μ exists due to the Kato-Rellich relative bound). To verify it one writes

$$\begin{aligned}
\frac{1}{H_n + V + \imath\mu} - P\frac{1}{\hat{H} + V_P + \imath\mu} &= \left(\frac{1}{H_n + \imath\mu} - P\frac{1}{\hat{H} + \imath\mu}\right) \\
&\quad - \left(\frac{1}{H_n + \imath\mu}V\frac{1}{H_n + V + \imath\mu} - P\frac{1}{\hat{H} + \imath\mu}V_P\frac{1}{\hat{H} + V_P + \imath\mu}\right) \\
&\quad + \left(\frac{1}{H_n + \imath\mu}VP\frac{1}{\hat{H} + V_P + \imath\mu} - \frac{1}{H_n + \imath\mu}VP\frac{1}{\hat{H} + V_P + \imath\mu}\right) \\
&= \left(\frac{1}{H_n + \imath\mu} - P\frac{1}{\hat{H} + \imath\mu}\right) \\
&\quad - \frac{1}{H_n + \imath\mu}V\left(\frac{1}{H_n + V + \imath\mu} - P\frac{1}{\hat{H} + V_P + \imath\mu}\right) \\
&\quad - \left(\frac{1}{H_n + \imath\mu} - P\frac{1}{\hat{H} + \imath\mu}\right)VP\frac{1}{\hat{H} + V_P + \imath\mu}
\end{aligned}$$

where we used that VP coincides with V_P on domains of $\hat{H}$ and $\hat{H} + V_P$ (i.e. on the range of the resolvents) and also that P commutes with the resolvents of $\hat{H}$ and $\hat{H} + V_P$. One can now solve for $\frac{1}{H_n+V+\imath\mu} - P\frac{1}{H_n+V_P+\imath\mu}$ which gives the identity above. □

Proposition 6.3.4 *Let ∇^2 be the one-dimensional Laplacian as a self-adjoint densely defined operator on $L^2(\mathbb{R})$. Let $\Theta = \chi_{\mathbb{R}_-}$ or let $\Theta \in C^\infty(\mathbb{R})$ be a monotonous function such that $\Theta|_{\mathbb{R}_+} = 0$ and $\Theta(x) = 1$ for all $x \leq 1$. Let Θ_n be the multiplication operator with $\Theta(n^2\cdot)$ then*

$$-\Delta + n\Theta_n \to -\Delta_+$$

in modified norm-resolvent sense where Δ_+ is the Laplacian on the positive half-line with Dirichlet boundary conditions

$$\mathrm{Dom}(\Delta_+) = \{\phi \in W_2^2(\mathbb{R}) : \phi(0) = 0\}.$$

Proof. It is shown in [66, Proposition 3.2] that for $\Theta = \chi_{\mathbb{R}_-}$ an actual (discontinuous) indicator function one has the required norm-resolvent convergence. There are two ways to get from there to the statement for smooth functions: the first is to go through the proof and note that it also works for a monotonous smooth function with compactly supported derivative that is rescaled with n. The second is to note that due to similar resolvent manipultations as in the proof of Proposition 6.3.3 the Schatten-p-norm of

$$(-\Delta + n\chi_{\mathbb{R}_-} + \imath)^{-1} - (-\Delta + n\Theta_n + \imath)^{-1}$$

can be bounded by a constant times

$$\begin{aligned} \left\|(-\Delta + n\Theta_n + \imath)^{-1}(n\chi_{\mathbb{R}_-} - n\Theta_n)\right\|_p &\leq \left\|(-\Delta + \imath)^{-1}(n\chi_{\mathbb{R}_-} - n\Theta_n)\right\|_p \\ &\leq c\, \|f\|_{L^2(\mathbb{R})} \left\|n\chi_{\mathbb{R}_-} - n\Theta_n\right\|_{L^2(\mathbb{R})} \end{aligned}$$

with $f(\lambda) = (\lambda^2 + \imath)^{-1}$, where a applied the well-known inequality [115, Chapter 4] for the Hilbert-Schmidt norm of an operator of type $f(\nabla)g(X)$ (which can be proven exactly as the present Proposition 2.1.5). Due to the quadratic rescaling one has $\left\|n\chi_{\mathbb{R}_-} - n\Theta_n\right\|_{L^2(\mathbb{R})} \leq cn^{-\frac{1}{2}}$, hence convergence to 0 of the resolvent differences in the Hilbert-Schmidt-norm which is stronger than norm convergence. □

Corollary 6.3.5 ([66, Corollary 3.4]) *Let Δ be the Laplacian on $L^2(\mathbb{R}^d)$ and Θ_n as in Proposition 6.3.4, as a multiplication operator $(\Theta_n\phi)(x) = \Theta_n(x_d)\phi(x)$ acting only on one coordinate. Then*

$$-\Delta + n\Theta_n \to -\Delta_+$$

converges in modified norm-resolvent sense to the Dirichlet-Laplacian $-\Delta_+$ on the positive halfspace.

Proof. We can use a similar argument as in [66, Corollary 3.4] to reduce to the one-dimensional case. Via Fourier transform in the first $d-1$ components the problem is equivalent to the assertion that in one dimension

$$R(\mu, k, n) := \left\|(-\Delta + n\chi_{\mathbb{R}_-} + k^2 + \imath\mu)^{-1} - P(-\Delta_+ + k^2 + \imath\mu)^{-1}\right\|$$

converges to 0 uniformly in k^2 for any μ. Applying the resolvent identities from the proof of Proposition 6.3.3 with $V = k^2$ show that for $\mu^2 > r$ convergence is uniform for all k in the compact ball $B_r(0)$. That implies that for $\mu = 1$ convergence is also uniform in any ball since due to Lemma 6.3.2 one has

$$R(1, k, n) \leq c_\mu R(\mu, k, n)$$

with a universal constant c_μ independent of k and n. On the other hand, we have the trivial estimate

$$R(1, k, n) \leq 2\frac{1}{k^2}$$

since all operators are bounded from below by k^2. Since that converges to 0 with increasing k^2, locally uniform convergence already implies globally uniform convergence. □

We can now write down the first main result of this section:

Proposition 6.3.6 *Let H_m be a self-adjoint operator on $L^2(\mathbb{R}^d, \mathbb{C}_N)$ of the form*

$$H = (-\nabla^2)\Sigma + \imath A \cdot \nabla + V \tag{6.3.3}$$

with Σ a self-adjoint unitary matrix, self-adjoint matrices $(A_1, ..., A_d)$ and $V = V^*$ a multiplication operator by a bounded smooth matrix-function.

Then the Dirichlet-restriction $\hat{H}$ of H is self-adjoint on $L^2(\mathbb{R}^{d-1} \times \mathbb{R}_+) \otimes \mathbb{C}_N$ and is the modified norm-resolvent limit of

$$H_m = (-\nabla^2 + m\Theta_m)\Sigma + \imath A \cdot \nabla + V$$

for a family of functions Θ_m as in Proposition 6.3.4. Both H and $\hat{H}$ are strictly smooth w.r.t. the position operators $X_1, ..., X_d$.

Proof. Since $-\Delta + m\Theta_m$ converges in modified norm-resolvent sense, the same is true for the tensor product with Σ. The remaining terms are relatively bounded and hence the Kato-Rellich sum also converges in modified resolvent sense by Proposition 6.3.3.

For the strict smoothness one easily shows that Definition 1.4.11 is satisfied for $\mathcal{E}_X$ the Schwartz functions, since $[H, X_i]$ respectively $[\hat{H}, X]$ are first-order differential operators and $[X_i, [X_j, H]]$ constant matrices. All higher commutators vanish. □

If the potential comes from the covariant Hilbert space representation $V_\omega = \pi_\omega(v)$ of a function $v \in C(\Omega)$ then it is obvious that $H_\omega = (-\nabla^2)\Sigma + \imath A \cdot \nabla + V_\omega$ defines a family of operators that is strongly affiliated to $C(\mathbb{R}^d_{0,\Omega})$ (and is also strictly smooth in the appropriate sense). If V is instead of the form $\mathcal{P}_+(X_d)(V_+)_\omega + \mathcal{P}_-(X_d)(V_-)_\omega$ for two potentials $V_\pm \in C(\Omega)$ and switch functions $\mathcal{P}_\pm$ then H defines a family of operators strongly affiliated to a two-sided Toeplitz extension $\mathrm{T}(C(\mathbb{R}^d_{0,\Omega}), \mathbb{R}, \xi)$. The strong affiliation of the Dirichlet-Laplacian implies that $\hat{H}$ is also strongly affiliated to the halfspace algebra $\hat{\mathcal{A}}$ constructed from the Toeplitz extension in Section 5.3.2.

Lemma 6.3.7 *For H and $\hat{H}$ as in Proposition 6.3.6 affiliated to $\mathcal{A} = C(\mathbb{R}^d_{0,\Omega})$ respectively the halfspace algebra $\hat{\mathcal{A}}$ as constructed in Section 5.3.2, any $\mu \neq 0$, $k > 0$ one has constants such that*

$$\left\| P_x((H + \mu\imath)^{-1} - (\hat{H} + \mu\imath)^{-1}) P_y \right\|_{L^p(\mathcal{A} \rtimes \mathbb{R}^d)} \leq C_{p,k} \langle x \rangle^{-k} \langle y \rangle^{-k} \langle x - y \rangle^{-k} \quad (6.3.4)$$

for all $x, y \in \mathbb{Z}^{d-1} \times \mathbb{N}$ and $p \in (\frac{d}{2}, \infty]$. Here P_x is as above Definition 4.3.11.

Proof. The bound (6.3.4) is not difficult to establish for the Laplacian alone, e.g. by computation of Cayley transforms as in Section 5.3.2. That immediately implies the same bound for $H_0 = -\nabla^2 \Sigma$ since w.l.o.g. Σ is a diagonal matrix. We can add back in the lower order terms $H = H_0 + V$ and apply the resolvent identity

$$\begin{aligned} &\frac{1}{H_0 + V + \imath\mu} - P \frac{1}{\hat{H}_0 + \hat{V} + \imath\mu} \\ &= \left(1 + \frac{1}{H + \imath\mu} V\right)^{-1} \left(P \frac{1}{\hat{H}_0 + \imath\mu} - \frac{1}{H_0 + \imath\mu}\right)\left(\hat{V} \frac{1}{\hat{H}_0 + \hat{V} + \imath\mu} - 1\right). \end{aligned}$$

The outer factors are strictly smooth since one checks that $\frac{1}{H_0+\imath\mu}V$ and $\hat{V}\frac{1}{\hat{H}_0+\hat{V}+\imath\mu}$ are strictly smooth (and the inverse of a smooth element is also smooth). Since smooth elements have rapidly decaying matrix elements one concludes (6.3.4). □

Another viable approach to verify that bounded functions of bulk and halfspace operators differ only by boundary terms is to directly estimate the integral kernels of the relevant operators as in [123][86].

Next, we check the conditions for the general assumptions of Section 6.1. The only non-trivial conditions are that the bounded transforms of the bulk and halfspace Hamiltonians differ only by a rapidly decaying boundary term:

Lemma 6.3.8 *For H and $\hat{H}$ as in Lemma 6.3.7, any $k > 0$ and $p > \frac{d}{2}$ one has constants such that*

$$\left\|P_x(F(H) - F(\hat{H}))P_y\right\|_p \le C_{p,k}\langle x\rangle^{-k}\langle y\rangle^{-k}\langle x-y\rangle^{-k}$$

for all $x, y \in \mathbb{Z}^{d-1} \times \mathbb{N}$.

Proof. The statement is true for the Hamiltonian $H_0 = -\nabla^2\Sigma$ and we argue that it will still be true after addition of the remaining terms V. We have

$$F(H_0+V) - F(H_0) = \int_{\mathbb{C}} (\partial_{\overline{z}}F_K(z))\left(\frac{1}{H_0+V+z} - \frac{1}{H_0+z}\right)\mathrm{d}z \wedge \mathrm{d}\overline{z}$$

where the integral converges in operator-norm since V is bounded w.r.t. the fractional power of the Laplacian $(1+H_0^2)^{-\frac{1}{4}}$.

Since a similar formula holds for $F(\hat{H}_0+\hat{V}) - F(\hat{H}_0)$ we write

$$\begin{aligned}&(F(\hat{H}_0+\hat{V}) - F(\hat{H}_0)) - (F(H_0+V) - F(H_0))\\ =& \int_{\mathbb{C}} (\partial_{\overline{z}}F_K(z))\Big(\frac{1}{\hat{H}_0+z}\hat{V}\frac{1}{\hat{H}_0+\hat{V}+z} - \frac{1}{H_0+z}V\frac{1}{H_0+V+z}\Big)\mathrm{d}z \wedge \mathrm{d}\overline{z}.\end{aligned}$$

We note that V and $\hat{V}$ are restrictions of an operator $\tilde{V}$ which is defined on the union of their domains. Thus we can rewrite the resolvent difference as

$$\begin{aligned}&P\frac{1}{\hat{H}_0+z}\hat{V}\frac{1}{\hat{H}_0+\hat{V}+z}P - P\frac{1}{H_0+z}V\frac{1}{H_0+V+z}P\\ &= P\left(\frac{1}{\hat{H}_0+z} - \frac{1}{H_0+z}\right)\tilde{V}\frac{1}{\hat{H}_0+\hat{V}+z}P\end{aligned}$$

$$
\begin{aligned}
&+ P\frac{1}{H_0+z}\hat{V}\frac{1}{\hat{H}_0+\hat{V}+z}P \\
&- P\frac{1}{H_0+z}V\frac{1}{H_0+V+z}P \\
&= P\left(\frac{1}{\hat{H}_0+z}-\frac{1}{H_0+z}\right)\tilde{V}\frac{1}{\hat{H}_0+\hat{V}+z}P \\
&- P\frac{1}{H_0+z}V\left(\frac{1}{H_0+V+z}-\frac{1}{\hat{H}_0+\hat{V}+z}\right)P.
\end{aligned}
$$

Since the resolvent differences of H_0 respectively H_0+V are p-localized at the boundary by Lemma 6.3.7 and $\frac{1}{H_0+z}V$ respectively $\hat{V}\frac{1}{\hat{H}_0+\hat{V}+z}$ extend to smooth operators with rapidly decaying matrix elements, we conclude that the difference is also localized at the boundary, namely

$$
\left\|P_x(F(H_0+V)-F(\hat{H}_0+\hat{V}))P_y\right\|_p \leq C_{p,k}\langle x\rangle^{-k}\langle y\rangle^{-k}\langle x-y\rangle^{-k}.
$$

□

To apply the results of Section 6.2 to an interface model we also need to check the secondary smoothness condition:

Proposition 6.3.9 *Let H be a quadratic Hamiltonian as in Proposition 6.3.6. Then H is $\Pi(X_d)$-differentiable for any switch function with compactly supported derivative and*

$$
[\Pi(X_d),H](1+H^2)^{-\frac{1}{4}-\epsilon}
$$

is ∞-localized.

Proof. We need to check (i) to (iii) of Definition 1.4.11. Since $\Pi(X_d)$ is bounded and thus everywhere defined, (i) shrinks down to the requirement that $\Pi(X_d)$ preserves $\mathrm{Dom}(H)$ since Π is smooth and the domain a Sobolev space. Computing the commutator one has

$$
[\Pi(X_d),H](1+H^2)^{-\frac{1}{4}} = 2\sigma_n\Pi'(X_d)\nabla_d(1+H^2)^{-\frac{1}{4}} + \sum_{k=1}^{n-1} v_{k,d}\Pi'(X_d)(1+H^2)^{-\frac{1}{4}}
$$

which extends to a bounded operator since ∇_d is bounded relative to $(1+H^2)^{-\frac{1}{4}}$ (the latter can be written as $(1-\Delta)^{-\frac{1}{2}}$ times an invertible bounded operator).

The second order commutator is even simpler

$$[\Pi(X_d),[\Pi(X_d),H]] = 2\sigma_n \Pi''(X_d).$$

Since $(1+H^2)^{-\frac{1}{4}-\epsilon}$ is ∞-smooth and Π' compactly supported it is easy to see that $[\Pi(X_d),H](1+H^2)^{-\frac{1}{4}-\epsilon}$ must also be ∞-localized if the bounded extension of $\sigma_n \nabla_d (1+H^2)^{-\frac{1}{4}}$ is ∞-smooth. The latter follows from Lemma 1.4.15 since $\sigma_n \nabla_d [X_d, H](1+H^2)^{-\frac{1}{4}-\epsilon}$ differs from $[X_d,H](1+H^2)^{-\frac{1}{4}-\epsilon}$ only by an ∞-smooth operator. □

We can now give quadratic continuum models that are non-trivial examples for Theorem 6.1.8 (although slightly artificial). The chiral one-dimensional Hamiltonian

$$H = \begin{pmatrix} 0 & 0 & -\nabla_x^2 & \nu+\nabla_x \\ 0 & 0 & \nu+\nabla_x & -\nabla_x^2 \\ -\nabla_x^2 & \nu-\nabla_x & 0 & 0 \\ \nu-\nabla_x & -\nabla_x^2 & 0 & 0 \end{pmatrix} + V$$

is for $V=0$ and $\nu \neq 0$ gapped and has a non-trivial winding number. With Dirichlet boundary conditions it satisfies the conditions of Theorem 6.1.8 and therefore has 0 as an exact eigenvalue. The expectation is that a mobility gap is formed under addition of a (not too disruptive) random potential. The bulk invariant and the bulk-boundary correspondence with its exact zero-energy edge mode should be stable if V is chiral, in particular since the index will still be an integer under ergodic disorder.

In two dimensions we know that a continuous topological insulator without potential does not admit weak Chern numbers since the classes of (differences of) Fermi projections $K_i(C_0(\mathbb{R}^d)) = K_{d \bmod 2}(\mathbb{C})$ distinguish only the top Chern number. This changes when one allows band-touching points. For example, the two-dimensional Hamiltonian

$$\begin{pmatrix} 0 & 0 & -\nabla_x^2+(\imath\nabla_y-\mu)^2 & \nu+\nabla_x \\ 0 & 0 & \nu+\nabla_x & -\nabla_x^2+(\imath\nabla_y-1)^2 \\ -\nabla_x^2+(\imath\nabla_y-1)^2 & \nu-\nabla_x & 0 & 0 \\ \nu-\nabla_x & -\nabla_x^2+(\imath\nabla_y-\mu)^2 & 0 & 0 \end{pmatrix}$$

has two Dirac-points in momentum space at $(k_x,k_y) = (0, \mu \pm \sqrt{\nu})$. The weak Chern number in x-direction $\mathrm{Ch}_{\mathcal{T},x}(u_F)$ is equal to the projected distance between the nodes. With Dirichlet boundary conditions it will exhibit a flat band of zero-energy edge states.

For a bulk-interface model that satisfies the conditions of Section 6.2 one can as mentioned before in $d = 2$ use two regularized Dirac-Hamiltonians (4.3.9) of different mass. They are also expected to develop a mobility gap in the bulk for large disorder. In $d = 3$ one might want to also construct a quadratic model $H = -\nabla^2\Sigma + \imath A \cdot \nabla$ which has an even number of Weyl points and non-vanishing two-dimensional Chern numbers. However, there can be none that satisfy the conditions of Theorem 6.2.8 since for two bands to touch in only isolated points one band must escape in energy to $+\infty$ and the other to $-\infty$ if resolvent-affiliation is supposed to hold, hence such a Hamiltonian cannot be semibounded. The way out is to apply a periodic matrix-valued potential $H = -\nabla^2 + V$ which leads to a pseudogap with Weyl points. According to a recent paper [63] such points can be proven to occur in potentials with certain symmetries but constructing an explicit example is apparently difficult.

Appendix

A Functional calculus

Let us provide some details on the smooth functional calculus, based on what is conventionally called the Helffer-Sjostrand formula (going back to [67], though it was proven earlier in [47]). The presentation here is based on [41]. For a smooth function with compact support $g \in C_0^\infty(\mathbb{R})$ define the almost analytic extension

$$\tilde{g}_K(x + \imath y) = \sum_{n=0}^{K} g^{(n)}(x) \frac{(\imath y)^n}{n!} \chi(y)$$

for χ a smooth symmetric cutoff function such that $\chi(y) = 1$ for $|y| < 1$ and $\chi(y) = 0$ for $|y| > 2$. With the notation $\partial_{\overline{z}} = \partial_x + \imath \partial_y$ one has

$$\partial_{\overline{z}} \tilde{g}_K(x + \imath y) = \imath \sum_{n=0}^{K} g^{(n)}(x) \frac{(\imath y)^n}{n!} \chi'(y) + g^{(K+1)}(x) \frac{(\imath y)^K}{K!} \chi(y). \qquad \text{(A.1)}$$

In the functional calculus there is a family of norms that appear naturally:

Definition A.1 *Let $\mathcal{S}^\beta(\mathbb{R})$ with $-1 < \beta < \infty$ be the set of functions for which each of the semi-norms*

$$\|g\|_{\mathcal{S}_K^\beta} := \sum_{n=0}^{K+1} \int_{\mathbb{R}} \left| g^{(n)}(x) \right| \langle x \rangle^{n-1+\beta}$$

is finite.

The space $\mathcal{S}^\beta(\mathbb{R})$ is roughly the space of functions whose r-th derivative decays faster than $\langle x \rangle^{-r-\beta}$. Most importantly for us, the bounded transform $F(\lambda) = \lambda(1+\lambda)^{-\frac{1}{2}}$ is in $\mathcal{S}^\beta(\mathbb{R})$ for all $-1 < \beta < 0$. For rapidly decaying functions one has a norm-convergent resolvent calculus:

Theorem A.2 (Smooth functional calculus) *Let H be regular self-adjoint operator on a Hilbert module. If $g \in \mathcal{S}^\beta(\mathbb{R})$ for some $\beta > 0$ then for any $K > 0$ one can write*

$$g(H) = \frac{1}{2\pi}\int_{\mathbb{C}} (\partial_{\bar{z}}\tilde{g}_K)(z)\frac{1}{H-z}\mathrm{d}z \wedge \mathrm{d}\bar{z}$$

converges in operator-norm. The integral can be restricted to the support of $\tilde{g}_K$ (which is compact if g is compactly supported). Moreover, one has $\left|\partial_{\bar{z}}\tilde{g}_K(x+\imath y)\right| \leq C_K\,|y|^K$ for small y which makes the integral norm-convergent.

For functions which do not decay at infinity the formula still holds morally speaking, but one may need to use different notions of convergence, e.g. strong, strict or weak convergence. Often it is easier to use approximation by compactly supported functions g_n for which $g_n(H)$ converges strictly:

Lemma A.3 *For any $g \in \mathcal{S}^\beta(\mathbb{R})$ with $-1 < \beta < \infty$ there exists a sequence of functions $g_n \in C_c^\infty(\mathbb{R})$ such that $\|g - g_n\|_{\mathcal{S}_k^\beta} \to 0$ for each k and which converges to g uniformly if $\beta > 0$ respectively uniformly on each compact set if $\beta \leq 0$.*

Proof. Choose a compactly supported function φ which is equal to 1 on the interval $[-1,1]$ then it is not difficult to check that the sequence $g_n = g\varphi(n\cdot)$ has the stated properties. □

When doing norm estimates with the smooth functional calculus the following integrals appear naturally:

Lemma A.4 *For $g \in \mathcal{S}^\beta(\mathbb{R})$ and positive exponents γ, s one has*

$$\int_{\mathbb{C}} \left|(\partial_{\bar{z}}\tilde{g}_K)(z)\right| |\Im m\, z|^{-\gamma}\left(1 + \frac{\langle \Re e\, z\rangle}{|\Im m z|}\right)^s \langle x\rangle^{-\ell}\mathrm{d}z \wedge \mathrm{d}\bar{z} \leq C_{K,\gamma,s}\,\|g\|_{\mathcal{S}_K^{1-\gamma-\ell}}$$

provided $K > \gamma + s$ and $\gamma > \beta$.

Proof. With the characteristic functions of

$$U = \{(x,y) : \langle x\rangle < |y| < 2\langle x\rangle\}, \quad V = \{(x,y) : 0 < |y| < 2\langle x\rangle\}$$

one obtains from (A.1) (see the proof of [41, Lemma 2.2.1])

$$\left|(\partial_{\bar{z}}\tilde{g}_K)(x+\imath y)\right| \leq c\sum_{n=0}^{K}\left|g^{(n)}(x)\right|\frac{|y|^n}{n!}\langle x\rangle^{-1}\chi_U(z) + c\left|g^{(K+1)}(x)\right|\frac{|y|^K}{K!}\chi_V(y).$$

The y-integration of than can be performed as long as $K > \gamma + s$ and can be done analytically. After that step the integrand is then bounded by

$$\left|(\partial_{\bar{z}}\tilde{g}_K)(x + \imath y)\right| \leq \sum_{n=0}^{K+1} c_{K,n,\gamma,s} \left|g^{(n)}(x)\right| \langle x \rangle^{n-\gamma-\ell}$$

as expected. □

Using that strategy one can apply the Helffer-Sjostrand calculus for estimates of perturbations when one has additional factors of resolvents to aid convergence:

Proposition A.5 *Let H be regular self-adjoint, V a bounded self-adjoint operator and $g \in \mathcal{S}^{\beta}(\mathbb{R})$ for some $\beta > -1$. Then for any $K > 0$*

$$g(H + V) - g(H) = \frac{1}{2\pi} \int_{\mathbb{C}} (\partial_{\bar{z}}\tilde{g}_K)(z) \frac{1}{H - z} V \frac{1}{H + V - z} \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

with a norm-convergent integral if $K > 2$.

If a is a bounded self-adjoint operator which preserves $\mathrm{Dom}(H)$ and for which $[H, a]$ extends to a bounded operator then

$$[g(H), a] = \frac{1}{2\pi} \int_{\mathbb{C}} (\partial_{\bar{z}}\tilde{g}_K)(z) \frac{1}{H - z} [a, H] \frac{1}{H - z} \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

with a norm-convergent integral if $K > 2$.

Proof. The first formula is true for all $g \in C_c^{\infty}(\mathbb{R})$ due to the resolvent identity

$$(H + z)^{-1} - (H + V + z)^{-1} = (H + V + z)^{-1} V (H + z)^{-1}$$

and the integral is absolutely convergent norm bounded by a universal constant times $\|g\|_{\mathcal{S}_K^{-1}} \|V\|$, hence it extends by continuity. The same reasoning passes for the second upon noting the identity

$$[(H + z)^{-1}, a] = (H + z)^{-1} [a, H] (H + z)^{-1}.$$

□

We can also use holomorphic functional calculus to approximate spectral projections:

Lemma A.6 ([111, Lemma 5.3.6]) *Let h be a bounded self-adjoint operator and $\mathcal{C}_\epsilon^\pm$ be the piecewise-linear contour in $\mathbb{C}$ which lies to the left respectively to the right of the spectrum $\sigma(h)$ and successively connects the points*

$$(\imath\epsilon, \imath, \imath \pm (\|h\| + 1), -\imath \pm (\|h\| + 1), -\imath, -\imath\epsilon).$$

Setting

$$\chi_\epsilon(h) \;=\; \frac{1}{2} - \frac{1}{\pi}\arctan\left(\frac{1}{\epsilon}\,h\right)\,, \qquad \mathrm{sgn}_\epsilon(h) \;=\; \frac{2}{\pi}\arctan\left(\frac{1}{\epsilon}\,h\right)\,,$$

one has

$$\chi_\epsilon(h) \;=\; \frac{-1}{2\pi\imath}\int_{\mathcal{C}_\epsilon^-} \frac{1}{h-z}\,\mathrm{d}z\,, \qquad \mathrm{sgn}_\epsilon(h) \;=\; \frac{1}{2\pi\imath}\int_{\mathcal{C}_\epsilon} \frac{1}{h-z}\,\mathrm{d}z$$

with the sum of 1-chains $\mathcal{C}_\epsilon = \mathcal{C}_\epsilon^+ + \mathcal{C}_\epsilon^-$. Hence

$$\underset{\epsilon\downarrow 0}{\text{s-lim}}\; \chi_\epsilon(h) \;=\; \chi(h<0) \;+\; \frac{1}{2}\,\chi(h=0)\,, \qquad \underset{\epsilon\downarrow 0}{\text{s-lim}}\; \mathrm{sgn}_\epsilon(h) \;=\; \mathrm{sgn}(h)$$

since Borel functional calculus maps pointwise convergent sequences to strong-operator convergent sequences.

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