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Black hole entropy in loop quantum gravity

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Abstract. We discuss the recent progress on black hole entropy in loop quantum gravity, focusing in particular on the recently discovered discretization effect for microscopic black holes. Powerful analytical techniques have been developed to perform the exact computation of entropy. A statistical analysis of the structures responsible for this effect shows its progressive damping and eventual disappearance as one increases the considered horizon area.

1. Introduction

The description of black hole entropy [1] is one of the most successful results of loop quantum gravity (LQG). The framework is based on the quantization of a spacetime containing an isolated horizon as an internal boundary. Upon quantization, the horizon is described by a $U(1)$ Chern-Simons theory of level k on a punctured sphere. The N distinguishable punctures concentrate the horizon degrees of freedom, codified in integer (mod k) numbers $a_i,\ i = 1, \ldots, N$ quantifying the distributional curvature. The spacetime surrounding the horizon is described, as usual in LQG, by spin-networks. The punctures on the horizon are created by those spin-network edges that pierce it. These edges carry two labels, spins $j_i \in \mathbb{N}/2$ and magnetic numbers $m_i \in \{-j_i, -j_i + 1, \ldots, j_i\},\ i = 1, \ldots, N$, and endow the horizon with an area given by

$$A_H = 8\pi\gamma\ell_P^2 \sum_{i=1}^{N} \sqrt{j_i(j_i+1)}, \qquad (1)$$

where γ is the so-called Barbero-Immirzi parameter.

The promotion of the isolated horizon boundary conditions to a quantum operator equation implies a relationship between a_i and m_i labels, namely

$$2m_i = -a_i \mod k, \qquad i = 1, \ldots, N. \tag{2}$$

Additionally, the spherical topology of (the spacial sections of) the horizon implies a constraint on the horizon labels a_i, a quantum version of the Gauss-Bonet theorem:

$$\sum_{i=1}^{N} a_i = 0. \tag{3}$$

These three constraints give rise to a precise combinatorial problem, namely to count all possible ordered lists of integer numbers a_i satisfying (3) and compatible through (2) and (1) with a given value of the horizon area $A \pm \delta A$, (the logarithm of) whose solution yields the entropy of a black hole as described by LQG.

The precise combinatorial problem was posed in [2] and its asymptotic solution (in the large area limit) was found in [3] to be given by

$$S(A) = \frac{\gamma_0}{\gamma} \frac{A}{4\ell_P^2} - \frac{1}{2} \ln \frac{A}{\ell_P^2} + O(A^0), \tag{4}$$

where γ_0 is a constant of known fixed value.

On the other hand, an exact computation in the low area regime [4] showed an effective discretization of entropy as a function of area for microscopic black holes, as shown in Figures 1 and 2.

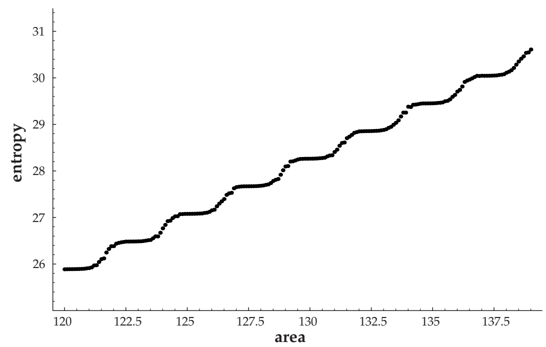

Figure 1. The entropy is plotted as a function of the horizon area in Plank units. The effective discretization effect is observed. The smearing interval δA is responsible for smoothing the band structure observed in Figure 2.

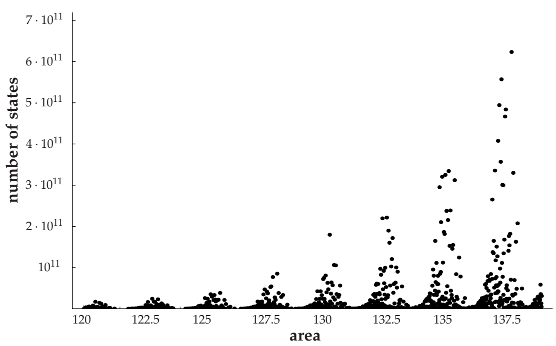

Figure 2. Degeneracy spectrum. The number of quantum states corresponding to each eigenvalue of the area operator is plotted. No smearing interval is considered. The periodic structure observed is responsible for the entropy discretization.

The question we are going to address here is whether these two results are compatible and what is the intermediate behavior in the area region between these two very different regimes.

2. Exact computational techniques

The first step toward understanding the compatibility of the results obtained by the two distinct approaches to the problem was taken in [5], where it was shown that, although the solution given in [3] correctly captures the first order asymptotic behavior, it neglects a mathematical subtlety that, when properly accounted for, leaves room for an additional oscillatory behavior of the kind responsible for the discretization effect. This solves the incompatibility problem and raises the new question of whether or not the discrete behavior of entropy does indeed extend to the large area limit.

In order to answer this, a series of technical developments were necessary, with the aim of extending the exact computations to larger values of area, and to obtain analytic expressions that capture the discrete nature of the problem, allowing an asymptotic analysis of the discrete structures.

In [6], new exact techniques to solve the combinatorial problem were introduced, based on number-theoretical strategies and the use of generating functions. The problem was divided in a four-step procedure:

(i) Characterization of the area spectrum (1) in terms of spins j, using the so-called Pell equation, that provided full control on the spin configurations compatible with every single area eigenvalue.
(ii) Implementation of the constraint (3) with techniques derived from the solution to the partition problem in number theory.
(iii) Computation of the degeneracy due to reordering of the labels, using standard combinatorics.
(iv) Summation of the results of the previous three steps for all the relevant values of area (i.e., introduction of the smearing interval δA), by means of an appropriately designed generating function.

These new techniques offer several advantages when studying the entropy computation. On the one hand, they allow to extend the exact computational results to larger values of area, as they can be efficiently implemented in a computational algorithm. On the other hand, they provide a structured understanding of the combinatorial problem, that allowed a thorough analysis [7] of many detailed features of the solution. In particular, the ability to separate the different sources of degeneracy in the spectrum and recognize the ones responsible for the discretization effect, and the possibility to find strategies to isolate partial sectors of the spectrum and study their particular properties, resulted of great interest to the analysis of the asymptotic behavior of entropy. Finally, the introduction of a solution fully based on generating functions [8] also provided analytical expressions that can be used as a departure point for asymptotic expansions.

Furthermore, these techniques have a wide range of applicability, and have been proven useful in related combinatorial problems, arising from different formulations of the black hole framework in LQG. It is particularly remarkable that in [9], by using the same techniques, it was possible to perform the entropy counting for the recently developed framework based on a $SU(2)$ invariant quantization of the isolated horizon [10].

3. The asymptotic behavior of entropy

Using the above presented techniques as a starting point, it was finally possible to answer the question about the asymptotic behavior of entropy in [11]. Following the introduction in [12] of a 'peak selector' variable $p \in \mathbb{N}$ (a particular combination of the number of punctures and the total sum of spins j on the horizon), it was possible to define a generating function that isolates each individual discrete structure (peak of degeneracy) in the black hole degeneracy spectrum (Figure 2):

$$G(\nu, z; x_1, x_2, \ldots) := \left(1 - \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} (z^{k_n^i} + z^{-z_n^i}) \nu^{3k_n^i+2} x_i^{y_n^i}\right)^{-1} . \tag{5}$$

The pairs of numbers $\{k_n^i, y_n^i\}$ are obtained from the solutions to the Pell equation. The coefficients in the series expansion of $G(\nu, z; x_1, x_2, \ldots)$ of terms with powers p of the variable ν, correspond to points within a single peak of degeneracy in the black hole degeneracy spectrum. Therefore, for each given value of the peak selector p, this generating function produces a fraction of the spectrum corresponding to a single peak of degeneracy.

The statistical properties of this generating function, and thus of the shape of the peaks, were studied using techniques from analytic combinatorics. In particular, it was possible to show that the distribution generated by (5) tends to a Gaussian as the peak selector p – and therefore the area of the horizon – tends to infinity. Furthermore, the asymptotic values of the mean μ_p and variance σ_p^2 corresponding to that Gaussian can be computed as a function of p. The result is given by the expressions:

$$\mu_p = (0.34959022\ldots) \cdot p, \quad \sigma_p^2 = (0.00009817\ldots) \cdot p \tag{6}$$

Therefore, though very slowly compared to the separation between consecutive peaks, the variance grows linearly with area, making the peak structures grow wider as the area tends to infinity. The consequence of this is that, progressively, consecutive peaks will start to overlap each other, smoothing out the discrete behavior of entropy and eventually washing it out completely. This prediction was tested in [11] by constructing a model based on the combination of all these consecutive Gaussians to obtain the behavior of entropy. The model showed the progressive damping of the staircase structure for increasing areas.

There is a caveat to that analysis, though. The value of the slope obtained for the asymptotic curve with the Gaussian model is slightly different (disagreement starting at the sixth significant figure) than the one obtained by the standard asymptotic computation (4). This implies that, although the agreement is extremely good for low values of area, there is an exponential divergence between the Gaussian model and the actual entropy in the asymptotic limit. The agreement is good enough for the disappearance of the discrete behavior to be observed at areas much lower than the ones for which the model starts to significantly disagree with the complete calculation, so the conclusions reached above stand. However, one cannot rule out a possible revival of the discretization effect (or some related kind of oscillatory behavior) for larger areas.

4. Conclusions

We have reviewed some of the latest contributions to the computation of black hole entropy within loop quantum gravity. The discretization effect for microscopic black holes is a purely quantum effect, and it is fully compatible with the linear asymptotic behavior predicted in the large area limit. The transition between these two distinct regimes occurs by a progressive smoothing of the staircase structure when the area grows out of the deep Planck regime.

Acknowledgments

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