

Pryms of non-cyclic triple coverings and log canonical models of the spin moduli space of genus 2

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ABSTRACT

We show that there exists a birational isomorphism between the moduli space of non-cyclic triple covers of genus-2 curves and that of even spin curves of genus 2. We describe the log canonical models of the moduli space S_2^+ and use this to extend the above birational map.

1. Introduction

In recent years, the log minimal model program has been carried out completely for the moduli space \overline{M}_g in low genera and in many cases these models have also a modular meaning. For instance, in [14] the Deligne–Mumford compactification and the geometric invariant theory compactification of M_2 are realized as log minimal models of \overline{M}_2 . In this paper, we study the log canonical models of the moduli space \overline{S}_2^+ of even stable spin curves of genus 2. Our initial motivation was to find out if the birational morphism, induced by the Prym map, between the moduli space $R_{2,3}^{nc}$ of non-cyclic étale triple coverings of genus-2 curves and S_2^+ (see [20, 21]) could be related to a new modular compactification of the spin moduli space. In order to state our theorems, let us recall some previous results.

Let $f : Y \rightarrow X$ be a non-cyclic étale threefold covering of a smooth projective curve X of genus 2. Its Prym variety $P(f) := (\text{Ker Nm } f)^0$ is a Jacobian surface with principal polarization Ξ , giving rise to a map $\text{Pr} : R_{2,3}^{nc} \rightarrow \mathcal{J}_2$ into the moduli space of canonically polarized Jacobian surfaces. In [20], we showed that this map is finite of degree 10 onto its image which is an open set in \mathcal{J}_2 . We also proved that the Prym map is not surjective and determined its image explicitly.

In [21], we extended this map to a proper map which is surjective onto the moduli space \mathcal{A}_2 of principally polarized abelian surfaces. To be more precise, we denote by $\mathfrak{S}_3 := \langle \sigma, \tau \mid \sigma^3 = \tau^2 = (\sigma\tau)^2 = 1 \rangle$ the symmetric group of order 6 and let ${}_{\mathfrak{S}_3}\overline{M}_2$ denote the moduli space of admissible \mathfrak{S}_3 -covers of stable curves of genus 2, as defined in [1]. We consider the following open subspace:

$${}_{\mathfrak{S}_3}\widetilde{M}_2 := \left\{ [h : Z \rightarrow X] \in {}_{\mathfrak{S}_3}\overline{M}_2 \mid \begin{array}{l} p_a(Z) = 7 \text{ and for any node } z \in Z, \\ \text{the stabilizer } \text{Stab}(z) \text{ is of order 3} \end{array} \right\}.$$

For all $[h : Z \rightarrow X] \in {}_{\mathfrak{S}_3}\widetilde{M}_2$, the Prym variety of $f : Y := Z/\langle \tau \rangle \rightarrow X$ is a principally polarized abelian surface, independent of the involution $\tau \in \mathfrak{S}_3$. This defines a Prym map $\text{Pr} : {}_{\mathfrak{S}_3}\widetilde{M}_2 \rightarrow \mathcal{A}_2$ which is proper, surjective and finite of degree 10.

The description of the fibres of the Prym map suggests that there is a close relationship between the space $\mathfrak{S}_3\widetilde{M}_2$ and the moduli space S_2^+ of even spin curves of genus 2. We define the following open subspace:

$$\mathfrak{S}_3\widehat{M}_2 := \{[h : Z \rightarrow X] \in \mathfrak{S}_3\widetilde{M}_2 \mid X \text{ irreducible}\}.$$

Then our first result is the following theorem.

THEOREM 1.1. *There is a canonical isomorphism $\alpha : \mathfrak{S}_3\widehat{M}_2 \rightarrow S_2^+$.*

This is true even on the level of moduli stacks (see Theorem 3.1). We then consider the question whether one can extend this isomorphism to a regular map defined on the whole of $\mathfrak{S}_3\widetilde{M}_2$ with values in some compactification of S_2^+ . For this, we work out the log canonical models of the compactification \overline{S}_2^+ of S_2^+ , as defined by Cornalba [8]. We denote by $\overline{\mathcal{S}}_2^+$ (respectively, \mathcal{S}_2^+) the moduli stack corresponding to \overline{S}_2^+ (respectively, S_2^+) and by $\delta := \overline{\mathcal{S}}_2^+ \setminus \mathcal{S}_2^+$ the boundary divisor. The *log canonical models* of $\overline{\mathcal{S}}_2^+$ with respect to $K_{\overline{\mathcal{S}}_2^+} + \epsilon\delta$ are defined by

$$\overline{\mathcal{S}}_2^+(\epsilon) := \text{Proj} \left(\bigoplus_{n \geq 0} \Gamma(\overline{\mathcal{S}}_2^+, n(K_{\overline{\mathcal{S}}_2^+} + \epsilon\delta)) \right),$$

for $\epsilon \in \mathbb{Q} \cap [0, 1]$. The fact that sections of invertible sheaves on $\overline{\mathcal{S}}_2^+$ are pullbacks of sections of the corresponding sheaves on \overline{S}_2^+ implies that the log canonical model of $\overline{\mathcal{S}}_2^+$ with respect to $K_{\overline{\mathcal{S}}_2^+} + \epsilon\delta$ can be identified with the log canonical model $\overline{\mathcal{S}}_2^+(\epsilon)$ of \overline{S}_2^+ with respect to the corresponding divisor in \overline{S}_2^+ (see Corollary 6.4).

The following theorem (see the end of Sections 6 and 9) is analogous to the corresponding result of Hassett's for \overline{M}_2 (see [14, Theorem 4.10]).

THEOREM 1.2. *Consider the log canonical model of \overline{S}_2^+ with respect to $K_{\overline{S}_2^+} + \epsilon\delta$, that is, the log canonical model of \overline{S}_2^+ with respect to the corresponding divisor in \overline{S}_2^+ .*

- (1) For $\epsilon > \frac{57}{25}$, we have $\overline{\mathcal{S}}_2^+(\epsilon) \simeq \overline{S}_2^+$.
- (2) For $\frac{49}{25} < \epsilon \leq \frac{57}{25}$, we have $\overline{\mathcal{S}}_2^+(\epsilon) \simeq \overline{S}_2^{+\text{inv}}$, where $\overline{S}_2^{+\text{inv}}$ denotes the invariant-theoretical compactification (for the definition see Section 7).
- (3) For $\epsilon = \frac{49}{25}$, we get a point; the log canonical divisor fails to be effective for $\epsilon < \frac{49}{25}$.

The invariant-theoretical compactification of S_2^+ is the Proj of a certain ring of invariants arising from the parametrization of binary sextic forms, together with a partition of the roots into two sets of three elements. Concerning the extension of the map α to the whole of $\mathfrak{S}_3\widetilde{M}_2$, we finally obtain the following theorem (see Propositions 4.2 and 10.1).

THEOREM 1.3. (1) *The construction defining the isomorphism $\alpha : \mathfrak{S}_3\widehat{M}_2 \rightarrow S_2^+$ does not extend to $\mathfrak{S}_3\widetilde{M}_2$.*

- (2) *The isomorphism $\alpha : \mathfrak{S}_3\widehat{M}_2 \rightarrow S_2^+$ extends to a regular map $\mathfrak{S}_3\widetilde{M}_2 \rightarrow \overline{S}_2^{+\text{inv}}$.*

The paper is organized as follows. In Section 2, we recall the definitions of the main moduli stacks and spaces which are used in the sequel. In Section 2, the proof of Theorem 1.1 is given. In Section 4, we show Theorem 1.3(1). In Section 5, we compute the nef cone of \overline{S}_2^+ and apply

this in Section 6 to prove Theorem 1.2(1). In Sections 7 and 8, we study the invariant-theoretical compactification \overline{S}_2^{+inv} and the canonical map $\overline{S}_2^+ \rightarrow \overline{S}_2^{+inv}$, and use this in Section 9 to give a proof of Theorem 1.2(2) and (3). Finally, in Section 10 we prove Theorem 1.3(2).

We work over an algebraically closed field k of characteristic zero. Moduli spaces are denoted by capital bold letters, the corresponding moduli stacks by the corresponding cursive letters. Divisors on a coarse moduli space are denoted by capital Latin letters, divisors on a moduli stack by small Greek letters. We denote a divisor and its class in the rational Picard group by the same letter.

2. The moduli stacks

2.1. The stacks of \mathfrak{S}_3 -coverings of genus-2 curves

Let $\mathfrak{S}_3\mathcal{M}_2$ denote the moduli stack of étale Galois covers of smooth curves of genus 2 with Galois group the symmetric group \mathfrak{S}_3 of order 6 [2, Theorem 17.2.11] and its compactification $\mathfrak{S}_3\overline{\mathcal{M}}_2$ of $\mathfrak{S}_3\mathcal{M}_2$ by admissible \mathfrak{S}_3 -covers as constructed in [1] (see also [2, Chapter 17]).

We consider the following open substack $\mathfrak{S}_3\widehat{\mathcal{M}}_2$ of $\mathfrak{S}_3\overline{\mathcal{M}}_2$ associated with the functor $F : Sch/\mathbb{C} \rightarrow Ens$ defined by

$$S \mapsto \left\{ [h : \mathcal{Z} \rightarrow \mathcal{X} \text{ over } S] \in \mathfrak{S}_3\overline{\mathcal{M}}_2 \mid \begin{array}{l} \text{for all } s \in S, p_a(\mathcal{Z}_s) = 7, \mathcal{X}_s \text{ is irreducible,} \\ \text{for any node } z \in \mathcal{Z}_s, \text{Stab}(z) \text{ is of order 3} \end{array} \right\}.$$

Note that $\mathfrak{S}_3\widehat{\mathcal{M}}_2$ is a smooth Deligne–Mumford stack of dimension 3. The functor F admits a coarse moduli space denoted by $\mathfrak{S}_3\widehat{M}_2$.

According to [21], the stack admits a stratification

$$\mathfrak{S}_3\widehat{\mathcal{M}}_2 = \mathfrak{S}_3\mathcal{M}_2 \sqcup \mathcal{R}_2 \sqcup \mathcal{R}_1.$$

Here \mathcal{R}_2 (respectively, \mathcal{R}_1) denotes the locally closed substack of $\mathfrak{S}_3\widehat{\mathcal{M}}_2$, where \mathcal{X}_s admits exactly one node (respectively, two nodes) for all s . The index refers to the dimension of the substack. Similarly, there is a stratification $\mathfrak{S}_3\widehat{M}_2 = \mathfrak{S}_3M_2 \sqcup R_2 \sqcup R_1$ for the corresponding moduli spaces.

2.2. The stacks of genus-2 spin curves

Recall that a smooth spin curve of genus 2 is a pair (C, κ) , with C a smooth curve of genus 2 and κ a theta characteristic on C , that is, a line bundle on C whose square is the canonical bundle. This definition extends in the obvious way to families of spin curves. Let \mathcal{S}_2 denote the moduli stack of smooth spin curves of genus 2. It is associated with the functor $G : Sch/\mathbb{C} \rightarrow Ens$ defined by

$$S \mapsto \left\{ \text{pairs}(\mathcal{C} \rightarrow S, \kappa_S) \mid \begin{array}{l} \mathcal{C} \rightarrow S \text{ is a smooth curve of genus 2 over } S, \\ \kappa_S \text{ is a theta characteristic of } \mathcal{C} \text{ over } S \end{array} \right\}.$$

The moduli stack \mathcal{S}_2 is a smooth Deligne–Mumford stack which decomposes into two irreducible components \mathcal{S}_2^+ and \mathcal{S}_2^- depending on the parity of the theta characteristic. Thus a spin curve (C, κ) is in \mathcal{S}_2^+ (respectively, \mathcal{S}_2^-) if and only if $h^0(\kappa)$ is an even (respectively, odd) number. Moreover, the forgetful map $\mathcal{S}_2 \rightarrow \mathcal{M}_2$ onto the moduli stack \mathcal{M}_2 of (smooth) curves of genus 2 is of degree 10 over \mathcal{S}_2^+ (respectively, 6 over \mathcal{S}_2^-). The functor G admits a coarse moduli space denoted by S_2 . Clearly, we have $S_2 = S_2^+ \sqcup S_2^-$.

We recall now the compactification $\overline{\mathcal{S}}_2$ of \mathcal{S}_2 constructed by Cornalba [8] (in fact, the construction works for arbitrary genus, we only need the genus-2 case). A rational component $E \subset X$ of a nodal curve X is called *exceptional* if $\#(E \cap \overline{X \setminus E}) = 2$. The curve X is called

quasi-stable if $\#(E \cap \overline{X \setminus E}) \geq 2$ for any smooth rational component $E \subset X$ and any two exceptional components are disjoint. A (generalized) spin curve of genus 2 is a triple (X, κ, β) , where X is a quasi-stable curve of (arithmetic) genus 2, $\kappa \in \text{Pic}^1(X)$ is a line bundle such that $\kappa_E = \mathcal{O}_E(1)$ for every exceptional component $E \subset X$ and $\beta : \kappa^{\otimes 2} \rightarrow \omega_X$ is a sheaf homomorphism which is generically non-zero along each non-exceptional component of X .

When X is smooth, κ is an ordinary theta characteristic and β is an isomorphism. A family of spin curves of genus 2 over a base scheme S consists of a triple $(\mathcal{X} \xrightarrow{f} S, \eta, \beta)$, where $f : \mathcal{X} \rightarrow S$ is a flat family of quasi-stable curves, $\eta \in \text{Pic}(\mathcal{X})$ is a line bundle and $\beta : \eta^{\otimes 2} \rightarrow \omega_{\mathcal{X}}$ is a sheaf homomorphism, such that at every point $s \in S$ the restriction $(\mathcal{X}_s, \eta_s, \beta_s)$ is a spin curve of genus 2. The compactification $\overline{\mathcal{S}}_2$ is the stack associated with the functor $G : S \mapsto \{(\mathcal{X} \xrightarrow{f} S, \eta, \beta)\}$. Again one has the obvious decomposition

$$\overline{\mathcal{S}}_2 = \overline{\mathcal{S}}_2^+ \sqcup \overline{\mathcal{S}}_2^-$$

depending on the parity of $h^0(C, \kappa)$.

Let us describe the boundary components of $\overline{\mathcal{S}}_2^+$, respectively, $\overline{\mathcal{S}}_2^-$. Denote by $\pi : \overline{\mathcal{S}}_2^+ \rightarrow \overline{M}_2$ the forgetful map $[X, \kappa, \beta] \mapsto [C]$, where C is the stable model of X is obtained from X by contracting all the exceptional components.

If $[X, \kappa, \beta] \in \pi^{-1}([X_1 \cup_y X_2])$, where X_1 and X_2 are elliptic curves intersecting transversally in a point y , then necessarily

$$X := X_1 \cup_{y_1} E \cup_{y_2} X_2,$$

where E is an exceptional component such that $X_1 \cap E = \{y_1\}$ and $X_2 \cap E = \{y_2\}$ with $\pi(y_i) = y$ for $i = 1$ and 2. Moreover,

$$\kappa = (\kappa_{X_1}, \kappa_{X_2}, \kappa_E = \mathcal{O}_E(1)) \in \text{Pic}^1(X),$$

with theta characteristics κ_{X_i} on X_i for $i = 1$ and 2. The condition $h^0(X, \kappa) \equiv 0 \pmod 2$ implies that κ_{X_1} and κ_{X_2} have the same parity. We denote

$$A_1 := \text{closure of } \left\{ (X, \kappa, \beta) \in \overline{\mathcal{S}}_2^+ \mid \begin{array}{l} X \text{ and } \kappa \text{ as above with } h^0(\kappa_{X_i}) = 0 \text{ for } i = 1, 2 \\ \text{and } \beta = \text{the obvious map (that is, zero on } E) \end{array} \right\}$$

and

$$B_1 := \text{closure of } \left\{ (X, \kappa, \beta) \in \overline{\mathcal{S}}_2^+ \mid \begin{array}{l} X \text{ and } \kappa \text{ as above with } \kappa_{X_i} = \mathcal{O}_{X_i} \text{ for } i = 1, 2 \\ \text{and } \beta = \text{the obvious map (that is, zero on } E) \end{array} \right\}.$$

If $[X, \kappa, \beta] \in \pi^{-1}([C])$ with C an irreducible one-nodal curve, $\nu : \tilde{C} \rightarrow C$ denotes the normalization of C , and

$$C = \tilde{C}/y_1 \sim y_2,$$

where y_1 and y_2 map to the node y of C , then there are two possibilities, namely

$$X = C \quad \text{or} \quad X = \tilde{C} \cup_{\{y_1, y_2\}} E,$$

with E an exceptional component. In the first case, let $\kappa_{\tilde{C}} := \nu^*(\kappa)$. Then $\kappa_{\tilde{C}}^{\otimes 2} = K_{\tilde{C}}(y_1 + y_2)$ and there is only one choice of gluing the fibres $\kappa_{\tilde{C}}(y_1)$ and $\kappa_{\tilde{C}}(y_2)$ to get κ such that $h^0(X, \kappa) \equiv 0 \pmod 2$. We denote

$$A_0 := \text{closure of } \{(X, \kappa, \beta) \in \overline{\mathcal{S}}_2^+ \mid X = C, \kappa \text{ as above and } \beta \text{ the obvious map}\}.$$

If $[X, \kappa, \beta] \in \pi^{-1}([C])$ with $X = \tilde{C} \cup_{\{y_1, y_2\}} E$, then $\kappa_{\tilde{C}} := \kappa \otimes \mathcal{O}_{\tilde{C}}$ is a theta characteristic on \tilde{C} and $\kappa|_E = \mathcal{O}_E(1)$. Since $H^0(X, \omega_X) \simeq H^0(\tilde{C}, \omega_{\tilde{C}})$, it follows that $\kappa_{\tilde{C}}$ is an even theta characteristic on the elliptic curve \tilde{C} . We denote

$$B_0 := \text{closure of } \left\{ (X, \kappa, \beta) \in \overline{\mathcal{S}}_2^+ \mid \begin{array}{l} X = \tilde{C} \cup_{\{y_1, y_2\}} E, \kappa \text{ as above} \\ \text{and } \beta \text{ the obvious map} \end{array} \right\}.$$

We denote by $\alpha_0, \beta_0, \alpha_1, \beta_1$ the corresponding divisors of the stack $\overline{\mathcal{S}}_2^+$. All in all, we have

$$\overline{\mathcal{S}}_2^+ = \mathcal{S}_2^+ \sqcup A_0 \sqcup B_0 \sqcup A_1 \sqcup B_1 \quad \text{and} \quad \overline{\mathcal{S}}_2^+ = \mathcal{S}_2^+ \sqcup \alpha_0 \sqcup \beta_0 \sqcup \alpha_1 \sqcup \beta_1.$$

3. The isomorphism $\mathfrak{S}_3\widehat{\mathcal{M}}_2 \rightarrow \mathcal{S}_2^+$

Let \mathcal{J}_2 be the open substack of the stack of canonically polarized Jacobians of smooth curves of genus 2. In [21], we showed that the Prym map

$$\text{Pr} : \mathfrak{S}_3\widehat{\mathcal{M}}_2 \rightarrow \mathcal{J}_2$$

is a finite surjective morphism of degree 10. The description of the fibres of Pr hints that there is a relation between the spaces $\mathfrak{S}_3\widehat{\mathcal{M}}_2$ and $\overline{\mathcal{S}}_2^+$. Note that the forgetful map $\mathcal{S}_2^+ \rightarrow \mathcal{M}_2$ is also of degree 10. The idea is, to associate with each admissible non-cyclic \mathfrak{S}_3 -covering $[Z \rightarrow X] \in \mathfrak{S}_3\widehat{\mathcal{M}}_2$, the theta divisor of the corresponding Prym variety $P(f)$ of the induced covering $f : Y = Z/\langle \tau \rangle \rightarrow X$, which in this case is a smooth genus-2 curve C , together with the theta characteristic arising from a naturally defined 6:1 map $C \rightarrow \mathbb{P}^1$.

THEOREM 3.1. *There is a canonical isomorphism of stacks*

$$\alpha : \mathfrak{S}_3\widehat{\mathcal{M}}_2 \rightarrow \mathcal{S}_2^+.$$

Proof. Let $[h : \mathcal{Z} \rightarrow \mathcal{X} \text{ over } S] \in \mathfrak{S}_3\widehat{\mathcal{M}}_2$. So the group $\mathfrak{S}_3 = \langle \sigma, \tau \rangle$ acts on \mathcal{Z} over S . The quotient $\mathcal{Y} := \mathcal{Z}/\langle \tau \rangle$ is an admissible cover of degree 3 of \mathcal{X} over S . Let

$$f : \mathcal{Y} \rightarrow \mathcal{X}$$

denote the induced morphism. With any closed point $s \in S$, we associated in [20, 21] a smooth curve $C(s)$ and a theta characteristic $\kappa(s)$ on $C(s)$ in the following way: Let $f(s) : \mathcal{Y}(s) \rightarrow \mathcal{X}(s)$ denote the normalization of the map $f(s) : \mathcal{Y}(s) \rightarrow \mathcal{X}(s)$. The curve $\mathcal{Y}(s)$ is hyperelliptic of genus 4 (respectively, 3 or 2 if $\mathcal{X}(s)$ has 1 or 2 nodes) and there is a commutative diagram (see [20, Proof of Theorem 5.1] in the smooth case and [21, Proof of Theorem 9.1] in the nodal case)

$$\begin{array}{ccc} \widetilde{\mathcal{Y}(s)} & \longrightarrow & \mathbb{P}^1 \xleftarrow{\varphi(s)} C(s) \\ \widetilde{f(s)} \downarrow & & \downarrow \bar{f}(s) \swarrow \psi(s) \\ \widetilde{\mathcal{X}(s)} & \longrightarrow & \mathbb{P}^1 \end{array} \tag{3.1}$$

where the horizontal maps of the square are the hyperelliptic coverings (respectively, a suitable double cover in the nodal case) and $C(s)$ is a smooth curve of genus 2, which is a theta divisor of the principal polarization of the Prym variety of the covering $f(s)$, with hyperelliptic cover $\varphi(s)$. To be more precise, in any case ($\mathcal{X}(s)$ smooth or not) there are six Weierstrass points q_1, \dots, q_6 of $\widetilde{\mathcal{Y}(s)}$ and two ramification points p_1 and p_2 of the double cover $\widetilde{\mathcal{X}(s)} \rightarrow \mathbb{P}^1$ such that

$$f(s)(q_1) = f(s)(q_2) = f(s)(q_3) = p_1 \quad \text{and} \quad f(s)(q_4) = f(s)(q_5) = f(s)(q_6) = p_2.$$

We can then consider the Prym variety of $f(s)$ as a subvariety of $\text{Pic}^2(\widetilde{\mathcal{Y}(s)})$ and the symmetric product $\widetilde{\mathcal{Y}(s)}^{(2)}$ as an open subset of $\text{Pic}^2(\widetilde{\mathcal{Y}(s)})$ in the usual way. In the smooth case, the curve $C(s)$ is given as (see [20, Section 4.5])

$$C(s) = \{ \mathcal{O}_Y(y+z) \in \text{Pic}^2(Y) \mid f(y) = f(\iota_Y z), z \neq y \},$$

with the reduced subscheme structure, where we write f for $f(s)$ and ι_Y denotes the hyperelliptic involution of Y . In the nodal case, there is a slight modification of this definition for which we refer the reader to [21, Section 8]. With this notation, the curve $C(s)$ is in any case irreducible and smooth of genus 2. Its Weierstrass points are the six points of $C(s)$:

$$\omega_1 = [q_1 + q_2], \omega_2 = [q_1 + q_3], \omega_3 = [q_2 + q_3], \omega_4 = [q_4 + q_5], \omega_5 = [q_4 + q_6], \omega_6 = [q_5 + q_6]$$

(see [20, Proposition 4.18; 21, Remark 8.9]). The composed map $\psi(s)$ is the morphism given by the pencil generated by the two divisors $2\omega_1 + 2\omega_2 + 2\omega_3$ and $2\omega_4 + 2\omega_5 + 2\omega_6$. Clearly, the linearly equivalent divisors $\omega_1 + \omega_2 - \omega_3$ and $\omega_4 + \omega_5 - \omega_6$ of $C(s)$ define an even theta characteristic $\kappa(s)$ on $C(s)$ which is uniquely and canonically determined by the covering $f(s)$.

By construction, the pair $(C(s), \kappa(s))$ depends algebraically on s . Hence we get a family of pairs

$$\alpha(h : \mathcal{Z} \rightarrow \mathcal{X}) := (\mathcal{C}, \kappa) \in \mathcal{S}_2^+$$

over S with $\mathcal{C}_s = C(s)$ and similarly for κ . Clearly, the family is flat as a family of smooth curves.

Conversely, let $[(\mathcal{C}, \kappa) \text{ over } S] \in \mathcal{S}_2^+$. So for every closed point $s \in S$, the curve $C(s) := \mathcal{C}_s$ is a smooth curve of genus 2 and $\kappa(s) := \kappa_s$ an even theta characteristic on it. Recall that if $W = \{\omega_1, \dots, \omega_6\}$ are the Weierstrass points of $C(s)$, the even theta characteristics are of the form $\kappa(s) = \mathcal{O}_{C(s)}(\omega_i + \omega_j - \omega_k) \simeq \mathcal{O}_{C(s)}(\omega_l + \omega_m - \omega_n)$, where $\omega_i, \omega_j, \omega_k$ are different points of W and $\omega_l, \omega_m, \omega_n$ their complement in W . Hence $\kappa(s)$ determines a partition of W into two complementary subsets of three elements of W . We use the associated divisors $2\omega_i + 2\omega_j + 2\omega_k$ and $2\omega_l + 2\omega_m + 2\omega_n$ to define a degree 6 covering $\psi(s) : C(s) \rightarrow \mathbb{P}^1$, which certainly factorizes via the hyperelliptic covering $\varphi(s) : C(s) \rightarrow \mathbb{P}^1$ and a degree 3 covering $\bar{f}(s) : \mathbb{P}^1 \rightarrow \mathbb{P}^1$. So we get the right-hand triangle of diagram (3.1).

The map $\bar{f}(s)$ is certainly unramified at the six points $\bar{\omega}_i = \varphi(\omega_i)$, and maps $\bar{\omega}_i, \bar{\omega}_j$ and $\bar{\omega}_k$ to \bar{p}_1 and $\bar{\omega}_l, \bar{\omega}_m$ and $\bar{\omega}_n$ to \bar{p}_2 , say. According to the Hurwitz formula, $\bar{f}(s)$ is of ramification degree 4. Suppose first that $\bar{f}(s)$ is simply ramified. Let $\bar{\omega}_7, \dots, \bar{\omega}_{10}$ denote the ramification points and $\bar{p}_3, \dots, \bar{p}_6$ the corresponding branch points. Then define hyperelliptic curves $\mathcal{Y}(s)$ with ramification points over $\bar{\omega}_1, \dots, \bar{\omega}_{10}$ and $\mathcal{X}(s)$ with ramification points over $\bar{p}_1, \dots, \bar{p}_6$.

Now suppose that $\bar{f}(s)$ is simply ramified at $\bar{\omega}_7$ and $\bar{\omega}_8$, and doubly ramified at $\bar{\omega}_9$ and let \bar{p}_3, \bar{p}_4 as well as \bar{p}_5 the corresponding branch points. In this case, define hyperelliptic curves $\mathcal{Y}(s)$ with ramification points over $\bar{\omega}_1, \dots, \bar{\omega}_8$, and a node over $\bar{\omega}_9$ and a curve $\mathcal{X}(s)$ of genus 1 with ramification points over $\bar{p}_1, \dots, \bar{p}_4$ and a node over \bar{p}_5 .

Finally, suppose that $\bar{f}(s)$ is doubly ramified at $\bar{\omega}_7$ and $\bar{\omega}_8$, and let \bar{p}_3, \bar{p}_4 denote the corresponding branch points. Then define hyperelliptic curves $\mathcal{Y}(s)$ with ramification points over $\bar{\omega}_1, \dots, \bar{\omega}_6$, and nodes over $\bar{\omega}_7$ and $\bar{\omega}_8$, and a curve $\mathcal{X}(s)$ of genus 0 with ramification points over \bar{p}_1 and \bar{p}_2 and nodes over \bar{p}_3 and \bar{p}_4 .

Looking at the ramification, one immediately checks that in any case the map $\bar{f}(s)$ lifts to an admissible covering $f(s) : \mathcal{Y}(s) \rightarrow \mathcal{X}(s)$. So we obtain a commutative diagram, whose normalization is diagram (3.1). Moreover, it is clear from the construction that the diagram varies algebraically with $s \in S$. So we get a family $f : \mathcal{Y} \rightarrow \mathcal{X}$ over S . Since in any case, the coverings $f_s = f(s)$ are not Galois, the Galois closure $h : \mathcal{Z} \rightarrow \mathcal{X}$ of $f : \mathcal{Y} \rightarrow \mathcal{X}$ over S is an admissible \mathfrak{S}_3 -cover over S . Clearly, we obtain an element

$$\beta((\mathcal{C}, \kappa)) := [h : \mathcal{Z} \rightarrow \mathcal{X} \text{ over } S] \in \mathfrak{S}_3 \widehat{\mathcal{M}}_2$$

over S . It is easy to verify that β is inverse to α .

It remains to check that isomorphisms in the category $\mathfrak{S}_3 \widehat{\mathcal{M}}_2$ are mapped by α to isomorphisms in the category \mathcal{S}_2^+ and conversely. This follows from the fact that the maps α and β are determined completely by the Weierstrass points and the ramifications of the involved

coverings and, under isomorphisms, these are mapped to Weierstrass points and ramifications of the same type. \square

As mentioned above, both stacks $\mathfrak{S}_3\widehat{\mathcal{M}}_2$ and \mathcal{S}_2^+ admit coarse moduli spaces $\mathfrak{S}_3\widehat{M}_2$ and S_2^+ . Since an isomorphism of stacks induces an isomorphism of the associated coarse moduli spaces, we obtain as an immediate consequence the following corollary.

COROLLARY 3.2. *The isomorphism $\alpha : \mathfrak{S}_3\widehat{\mathcal{M}}_2 \rightarrow \mathcal{S}_2^+$ induces an isomorphism of moduli spaces $\mathfrak{S}_3\widehat{M}_2 \rightarrow S_2^+$, also denoted by α .*

4. Non-extension of the construction defining α

In [21], we considered the following subscheme of the moduli scheme $\mathfrak{S}_3\overline{M}_2$ of admissible \mathfrak{S}_3 -covers of stable curves of genus 2:

$$\mathfrak{S}_3\widetilde{M}_2 := \left\{ [h : Z \rightarrow X] \in \mathfrak{S}_3\overline{M}_2 \mid \begin{array}{l} p_a(Z) = 7 \text{ and for any node } z \in Z \\ \text{the stabilizer } \text{Stab}(z) \text{ is of order } 3 \end{array} \right\},$$

and we showed that

$$\mathfrak{S}_3\widetilde{M}_2 = \mathfrak{S}_3\widehat{M}_2 \sqcup S,$$

where S is the closed subscheme of $\mathfrak{S}_3\widetilde{M}_2$ consisting of reducible \mathfrak{S}_3 -coverings. In this section, we study the question whether the isomorphism $\alpha : \mathfrak{S}_3\widehat{M}_2 \rightarrow S_2^+$ extends to a morphism $\mathfrak{S}_3\widetilde{M}_2 \rightarrow \overline{S}_2^+$.

LEMMA 4.1. *There is a non-empty open set $U \subset S$ such that α extends to a holomorphic map $\mathfrak{S}_3\widehat{M}_2 \sqcup U \rightarrow \overline{S}_2^+$.*

Note that $\mathfrak{S}_3\widehat{M}_2 \sqcup U$ is open in $\mathfrak{S}_3\widetilde{M}_2$ with complement of codimension at least 2.

Proof. The moduli space $\mathfrak{S}_3\overline{M}_2$ of admissible G -covers of stable curves of genus 2 is a normal projective variety (see [2]). So the open subvariety $\mathfrak{S}_3\widetilde{M}_2$ is also normal. Hence its singular locus is of codimension at least 2. Now α defines a rational map $\bar{\alpha}$ from the smooth variety $(\mathfrak{S}_3\widetilde{M}_2)_{\text{reg}}$ to \overline{S}_2^+ . Since any such rational map extends to a holomorphic map in codimension 1 (see [13, p. 491]), this implies the assertion. \square

However, we have the following proposition.

PROPOSITION 4.2. *The construction giving the isomorphism $\alpha : \mathfrak{S}_3\widehat{M}_2 \rightarrow S_2^+$ does not extend to $\mathfrak{S}_3\widetilde{M}_2$.*

In order to see what we have to show, let us recall the definition of the map α in the smooth case. Let $Z \rightarrow X$ be an étale \mathfrak{S}_3 -covering of a smooth curve X of genus 2 and $f : Y \rightarrow X$ be a corresponding non-cyclic étale degree-3 covering. Let h_X denote the hyperelliptic line bundle on X , and Θ the canonical theta divisor in $\text{Pic}^3(Y)$ given by the image of the map $Y^{(3)} \rightarrow \text{Pic}^3(Y)$. If q is one of the two Weierstrass points of Y such that $f^{-1}(f(q))$ consists of three Weierstrass points, then we consider the following translate of Θ :

$$\Theta_q := \Theta - q \subset \text{Pic}^2(Y).$$

Let $Nm : \text{Pic}^2(Y) \rightarrow \text{Pic}^2(X)$ be the norm map. We define \tilde{P} as the following translation of the Prym variety P of f :

$$\tilde{P} := Nm^{-1}(h_X) \subset \text{Pic}^2(Y).$$

Then we have [20, Corollary 4.12]

$$\tilde{P} \cap \Theta_q = \Xi_1 \cup \Xi_2 \cup \Xi_3,$$

where the Ξ_i are algebraically equivalent divisors, each of them defining the principal polarization of \tilde{P} , and thus pairwise isomorphic smooth curves of genus 2. If $\alpha(f) = (C, \kappa)$, then by definition one has

$$C = \Xi_1.$$

For the proof of Proposition 4.2, the theta characteristic κ is not relevant.

Now, let $f : Y \rightarrow X$ be the non-cyclic degree-3 covering given by a general element of S . So $Y = Y_1 \cup_{y_0} Y_2$ and $X = X_1 \cup_{x_0} X_2$ with smooth curves Y_i of genus 2 (respectively, X_i of genus 1) intersecting transversally in the point y_0 (respectively, x_0) and $f = f_1 \cup_{y_0} f_2$ with non-cyclic degree-3 covers $f_i : Y_i \rightarrow X_i$ totally ramified in the point $y_{0i} (= y_0) \in Y_i$ for $i = 1, 2$ (we denote the point y_0 by y_{0i} when considered as a point of Y_i). In order to prove Proposition 4.2, we will show that, defining \tilde{P} and Θ_q in the same way as in the smooth case, the intersection $\tilde{P} \cap \Theta_q$ is not proper, that is, not a divisor in \tilde{P} . We need first some preliminaries.

Let q_i^0, \dots, q_i^5 denote the Weierstrass points of Y_i for $i = 1, 2$. Then y_0 is necessarily one of these points, say $y_0 = q_1^0 = q_2^0$. For $i = 1, 2$, the hyperelliptic involution of X_i lifts to an involution on Y_i , which induce an involution on Y . Thus we get the following commutative diagram:

$$\begin{CD} Y = Y_1 \cup_{y_0} Y_2 @>\gamma = \gamma_1 \cup_{y_0} \gamma_2>> \mathbb{P}^1 \cup_{\gamma(y_0)} \mathbb{P}^1 \\ @V f = f_1 \cup_{x_0} f_2 VV 3:1 @VV 3:1 V \bar{f} = \bar{f}_1 \cup_{\gamma(y_0)} \bar{f}_2 \\ X = X_1 \cup_{x_0} X_2 @>\delta = \delta_1 \cup_{x_0} \delta_2>> \mathbb{P}^1 \cup_{\delta(x_0)} \mathbb{P}^1 \end{CD} \tag{4.1}$$

where γ_i and δ_i are the hyperelliptic coverings. It follows that the image of any Weierstrass point under the map f_i is a ramification point of δ_i . Since δ_i admits four ramification points, say p_i^0, \dots, p_i^3 , we conclude that there is one point p_i^j such that $f_i^{-1}(p_i^j)$ consists of three Weierstrass points, and three points p_i^j such that $f_i^{-1}(p_i^j)$ contains only one Weierstrass point. Without loss of generality, we may assume that

$$f(q_i^1) = f(q_i^2) = f(q_i^3) = p_i^1$$

and

$$f(q_i^4) = p_i^2, \quad f(q_i^5) = p_i^3 \quad \text{and} \quad f(q_i^0) = p_i^0 = x_{0i}.$$

The normalization of Y (respectively, X) is given by $n_Y = \iota_{Y_1} \cup \iota_{Y_2}$ (respectively, $n_X = \iota_{X_1} \cup \iota_{X_2}$), where ι_{Y_i} (respectively, ι_{X_i}) denote the canonical embeddings $Y_i \rightarrow Y$ (respectively, $X_i \rightarrow X$). They induce canonical isomorphisms of the Picard varieties

$$n_Y^* : \text{Pic}(Y) \rightarrow \text{Pic}(Y_1) \times \text{Pic}(Y_2) \quad \text{and} \quad n_X^* : \text{Pic}(X) \rightarrow \text{Pic}(X_1) \times \text{Pic}(X_2). \tag{4.2}$$

In the sequel, we identify both sides, that is, denote the elements of $\text{Pic}(Y)$ and $\text{Pic}(X)$ by pairs (L_1, L_2) with $L_i \in \text{Pic}(Y_i)$ (respectively, $\text{Pic}(X_i)$).

Now $\text{Pic}^3(Y)$ consists of infinitely many components, however, there are only two components, namely $\text{Pic}^{(2,1)}(Y)$ and $\text{Pic}^{(1,2)}(Y)$ (with the obvious notation) which admit a canonical theta divisor (see [5, Proposition 2.2]); these are also the only balanced components in the sense of Caporaso’s compactified Picard varieties [6]. Since the situation is symmetric in Y_1 and Y_2 ,

we work only with $\text{Pic}^{(2,1)}(Y)$. Then the canonical theta divisor is

$$\Theta := \{L \in \text{Pic}^{(2,1)}(Y) = \text{Pic}^2(Y_1) \times \text{Pic}^1(Y_2) \mid h^0(L) \geq 1\},$$

with reduced subscheme structure. The following lemma is shown by using Riemann–Roch formula.

LEMMA 4.3. *With the identifications $Y_1 + y_{01} = \{\mathcal{O}_{Y_1}(y_1 + y_{01}) \mid y_1 \in Y_1\}$ and $Y_2 = \{\mathcal{O}_{Y_2}(y_2) \mid y_2 \in Y_2\}$, we have*

$$\Theta = [(Y_1 + y_{01}) \times \text{Pic}^1(Y_2)] \cup [\text{Pic}^2(Y_1) \times Y_2].$$

If a family of smooth \mathfrak{S}_3 -coverings degenerates to the map $f : Y \rightarrow X$ described above, then the Weierstrass point q chosen for the translate of Θ , specializes to one of the points q_1^i or q_2^i with $i = 1, 2$ or 3 . In the latter case, $\Theta - q_2^i$ is a divisor of $\text{Pic}^{(2,0)}$ which is not balanced. So we need to work with one of the q_1^i , say $q = q_1^1$ and define

$$\Theta_q := \Theta - q \in \text{Pic}^{(1,1)}(Y).$$

There is exactly one line bundle of degree 2 with $h^0 = 2$ in $\text{Pic}^{(1,1)}(Y)$, respectively, $\text{Pic}^{(1,1)}(X)$, namely

$$h_Y = (\mathcal{O}_{Y_1}(q_{01}), \mathcal{O}_{Y_2}(q_{02})), \quad \text{respectively,} \quad h_X = (\mathcal{O}_{X_1}(p_{01}), \mathcal{O}_{X_2}(p_{02})).$$

If $\text{Nm}_f : \text{Pic}^2(Y) \rightarrow \text{Pic}^2(X)$ denotes the norm map, then we define, as in the smooth case, \tilde{P} as the following translate of the Prym variety P of f :

$$\tilde{P} := \text{Nm}_f^{-1}(h_X) \subset \text{Pic}^{(1,1)}(Y).$$

LEMMA 4.4. *The intersection $\Theta_q \cap \tilde{P}$ contains the three pairwise disjoint curves*

$$\Sigma_i := \{(\mathcal{O}_{Y_1}(q_1^i + y_{01} - q), \mathcal{O}_{Y_2}(y_{02} + y_2 - y_2')) \mid y_2 \in Y_2, y_2' \in f_2^{-1}f_2(y_2)\},$$

for $i = 1, 2, 3$ which are $3 : 1$ -coverings of Y_2 .

Proof. According to Lemma 4.3, we have $\Sigma_i \subset \Theta_q$. By the identifications (4.2), $\text{Nm}_f = \text{Nm}_{f_1} \times \text{Nm}_{f_2}$ and one computes

$$\begin{aligned} &\text{Nm}_f((\mathcal{O}_{Y_1}(q_1^i + y_{01} - q), \mathcal{O}_{Y_2}(y_{02} + y_2 - \iota_{Y_2}(y_2))) \\ &= (\mathcal{O}_{X_1}(p_1 + x_{01} - p_1), \mathcal{O}_{X_2}(x_{02} + f_2(y_2) - f_2(y_2))) \\ &= (\mathcal{O}_{X_1}(x_{01}), \mathcal{O}_{X_2}(x_{02})) = h_X. \end{aligned}$$

So $\Sigma_i \subset \tilde{P}$. The curves Σ_i are pairwise disjoint, since $q = q_1^1$ and the line bundles $\mathcal{O}_{X_1}(q_1^i + y_{01} - q_1^1)$ are pairwise different. The last assertion follows from the fact that f_2 is a $3 : 1$ -covering, and hence there are three preimages $y_2' \in f_2^{-1}f_2(y_2)$. □

Proof of Proposition 4.2. We may consider $\text{Pic}^{(1,1)}(Y)$ as an abelian variety and \tilde{P} as an abelian subvariety, since both contain the distinguished point h_Y . According to [21, Proposition 5.3],

$$\tilde{P} = P_1 \times P_2,$$

with elliptic curves $P_i = \text{Prym}(f_i)$ for $i = 1, 2$ and canonical principal polarization. On the other hand, the principal polarization of $\text{Pic}^{(1,1)}(Y)$ is defined by the divisor Θ_q . So if the construction of α would extend to the closed subvariety S , the divisor Θ_q would restrict to

a divisor defining the threefold of the canonical principal polarization of \tilde{P} . Being pairwise disjoint, the curves Σ_i would define the canonical principal polarization, that is, would be isomorphic to $P_1 \times \{0\} \cup \{0\} \times P_2$. But this contradicts Lemma 4.4 since, the curve Y_2 being of genus 2, any 3:1 covering is of arithmetic genus greater than 2, and hence Σ_i cannot define a principal polarization. \square

5. The nef cone of \overline{S}_2^+

In this section, we compute the cone of numerically effective divisors of \overline{S}_2^+ in the rational Picard group $\text{Pic}_{\mathbb{Q}}(\overline{S}_2^+)$. For this, we consider the moduli space $\overline{M}_{0,[3,3]}$ of stable curves of genus 0 with six unordered marked points partitioned into two sets of three points. If $\overline{M}_{0,6}$ denotes the usual moduli space of stable 6-pointed curves of genus 2 and

$$G := (\mathfrak{S}_3 \times \mathfrak{S}_3) \rtimes \langle \tau \rangle,$$

where the first \mathfrak{S}_3 acts on the numbers 1, 2, 3, the second on the numbers 4, 5, 6 and $\tau = (14)(25)(36)$, then

$$\overline{M}_{0,[3,3]} = \overline{M}_{0,6}/G.$$

Here we consider the numbers as indices of the marked points. We use the fact (see [19, Lemma 20]) that there is a canonical isomorphism

$$\overline{M}_{0,[3,3]} \simeq \overline{S}_2^+.$$

In the sequel, we often identify both spaces and denote corresponding divisors by the same letter. Moreover, the composed map $\overline{M}_{0,6} \rightarrow \overline{S}_2^+$ maps the boundary of $\overline{M}_{0,6}$ onto the boundary of \overline{S}_2^+ (see [19]).

Recall that the boundary of $\overline{M}_{0,6}$ consists of divisors Δ_S , where $S \subset \{1, \dots, 6\}$ with $|S|, |S^c| \geq 2$. Each Δ_S is the closure of the points corresponding to reducible curves with one node and where S points are marked in one component. We denote the class of Δ_S in $\text{Pic}_{\mathbb{Q}}(\overline{M}_{0,6})$ by the same letter.

The divisors in $\overline{M}_{0,[3,3]}$ can be regarded as the divisors in $\overline{M}_{0,6}$ which are invariant under the action of G . The isomorphism $\overline{M}_{0,[3,3]} \simeq \overline{S}_2^+$ is defined in a natural way by associating to a rational curve with six marked points the admissible double covering ramified over those points [19]. Moreover, it maps the following G -invariant divisors into the boundary divisors of \overline{S}_2^+ (see [19, Table in Section 3.1]):

$$\Delta^{11} := \sum_{S \in \text{Orb}_G(12)} \Delta_S \mapsto A_0, \tag{5.1}$$

$$\Delta^{12} := \sum_{S \in \text{Orb}_G(14)} \Delta_S \mapsto B_0, \tag{5.2}$$

$$\Delta_{123}^c := \sum_{S \in \text{Orb}_G(124)} \Delta_S \mapsto A_1, \tag{5.3}$$

$$\Delta_{123} := \Delta_{\{1,2,3\}} \mapsto B_1. \tag{5.4}$$

In order to see the G -invariance of these divisors, use the fact that if $\{i_1, \dots, i_6\} = \{1, \dots, 6\}$, then $\Delta_{\{i_1, i_2, i_3\}} = \Delta_{\{i_4, i_5, i_6\}}$. When we consider the G -invariant classes $\Delta^{11}, \Delta^{12}, \Delta_{123}^c$ and Δ_{123} as elements of $\text{Pic}_{\mathbb{Q}}(\overline{M}_{0,[3,3]})$, we denote them, respectively, by A_0, B_0, A_1 and B_1 . According to [19], the classes A_0, B_0, A_1 and B_1 generate the \mathbb{Q} -vector space $\text{Pic}_{\mathbb{Q}}(\overline{M}_{0,[3,3]})$.

In order to compute the nef cone of $\overline{M}_{0,[3,3]}$, we introduce some F -curves. An F -curve in $\overline{M}_{0,6}$ is obtained as the image of the map $\nu : \overline{M}_{0,4} \rightarrow \overline{M}_{0,6}$ defined by attaching one 3-pointed

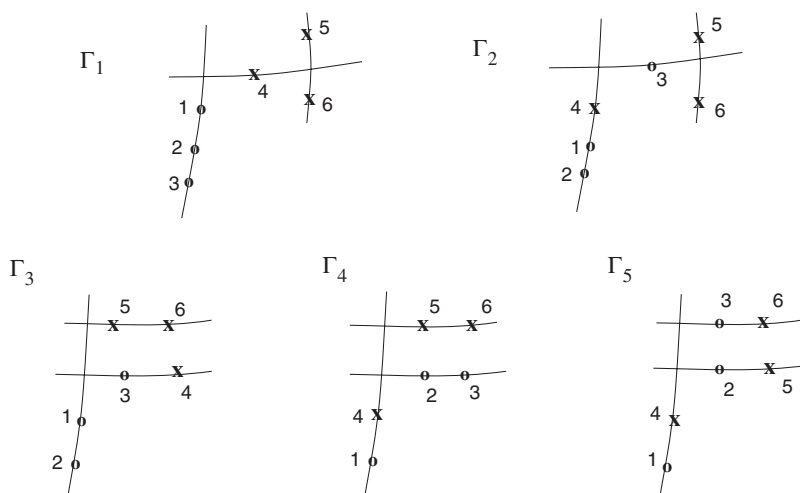


FIGURE 1. Generic points of the F -curves.

curve at one of the four marked points or two 2-pointed curves at two of the marked points [12, End of Section 2]. Note that, as a complete intersection curve in $\overline{M}_{0,6}$, the curve F always consists of three irreducible components, in the first case the attached curve is semistable with two irreducible components, whereas in the second case the two attached curves are irreducible.

We consider the following F -curves. Let Γ_1 (respectively, Γ_2) denote the curve in $\overline{M}_{0,6}$ whose elements are constructed by attaching to a spine labelled with the subset $\{1, 2, 3\}$ (respectively, $\{1, 2, 4\}$) of $\{1, \dots, 6\}$ to a fourth point a rational curve consisting of two components, where the component directly attached to the spine is labelled with 4 (respectively, 3) and the other curve with 5 and 6 (see Figure 1). This defines a curve in $\overline{M}_{0,6}$, since the fourth point of the spine moves freely, whereas the other components have three marked points: the nodes and the labelled points.

Let Γ_3 be the curve constructed by attaching two 2-pointed irreducible curves to the spine, where exactly one of the attaching points moves and the two points on the spine are labelled with $\{1, 2\}$. Similarly, we define Γ_4 (respectively, Γ_5) by attaching two 2-pointed curves, and where the labelled points on the spine are $\{1, 4\}$ and one tail is labelled with $\{2, 3\}$ (respectively, $\{2, 5\}$) (see Figure 1).

In principle, one should consider the orbit of an F -curve under the action of the group G , but in order to determine the inequalities defining the nef cone it will be sufficient to intersect the divisors with a representative of the orbit.

LEMMA 5.1. *A rational divisor class $aA_0 + bB_0 + cA_1 + dB_1$ of \overline{S}_2^+ is nef (respectively, ample) if and only if the corresponding class $D := a\Delta^{11} + b\Delta^{12} + c\Delta_{123}^c + d\Delta_{123}$ of $\overline{M}_{0,6}$ satisfies*

$$(D \cdot \Gamma_i) \geq 0 \quad (\text{respectively, } > 0),$$

for $i = 1, \dots, 5$.

Proof. The class of D being G -invariant, D is nef as a divisor of $\overline{M}_{0,[3,3]}$ if and only if it is nef as a divisor of $\overline{M}_{0,6}$. Hence according to [16, Theorem 1.3] D is nef if and only if it intersects any F -curve of $\overline{M}_{0,6}$ non-negatively. Since D is G -invariant, it suffices to check this

for a representative of the G -orbits of F -curves. Now it is easy to check that there are exactly six G -orbits with representatives $\Gamma_1, \dots, \Gamma_5$ and Γ'_2 . Here Γ'_2 differs from Γ_2 only by labelling the middle component with 5 (instead of 3) and the last component with 3 and 6 (instead of 5 and 6). However, by [16, Lemma 4.3] Γ'_2 is numerically equivalent to Γ_2 . So this implies the assertion on the nefness and also on the ampleness of the divisor class. \square

LEMMA 5.2. *The intersection numbers of the divisors $\Delta^{11}, \dots, \Delta_{123}$ with Γ_i are*

$$\begin{aligned} (\Delta^{11} \cdot \Gamma_1) &= 3, & (\Delta^{12} \cdot \Gamma_1) &= 0, & (\Delta_{123}^c \cdot \Gamma_1) &= 0, & (\Delta_{123} \cdot \Gamma_1) &= -1, \\ (\Delta^{11} \cdot \Gamma_2) &= 1, & (\Delta^{12} \cdot \Gamma_2) &= 2, & (\Delta_{123}^c \cdot \Gamma_2) &= -1, & (\Delta_{123} \cdot \Gamma_2) &= 0, \\ (\Delta^{11} \cdot \Gamma_3) &= 0, & (\Delta^{12} \cdot \Gamma_3) &= -1, & (\Delta_{123}^c \cdot \Gamma_3) &= 2, & (\Delta_{123} \cdot \Gamma_3) &= 0, \\ (\Delta^{11} \cdot \Gamma_4) &= -2, & (\Delta^{12} \cdot \Gamma_4) &= 1, & (\Delta_{123}^c \cdot \Gamma_4) &= 1, & (\Delta_{123} \cdot \Gamma_4) &= 1, \\ (\Delta^{11} \cdot \Gamma_5) &= 0, & (\Delta^{12} \cdot \Gamma_5) &= -1, & (\Delta_{123}^c \cdot \Gamma_5) &= 2, & (\Delta_{123} \cdot \Gamma_5) &= 0. \end{aligned}$$

Proof. This follows by an immediate computation using [16, Lemma 4.3], which gives the intersection numbers of any boundary divisor with any F -curve of $\overline{M}_{0,n}$. \square

From this, we immediately conclude the following criterion for a rational divisor class on $\overline{M}_{0,[3,3]} = \overline{S}_2^+$ to be nef or ample.

PROPOSITION 5.3. *A \mathbb{Q} -divisor $D \equiv aA_0 + bB_0 + cA_1 + dB_1$ is nef (respectively, ample) if and only if the following inequalities are satisfied:*

$$3a \geq d, \quad a + 2b \geq c, \quad 2c \geq b \quad \text{and} \quad b + c + d \geq 2a$$

(respectively, all inequalities are strict).

Proof. According to Lemma 5.2 and the identifications (5.1), ..., (5.4), we have

$$\begin{aligned} (D \cdot \Gamma_1) &= 3a - d, & (D \cdot \Gamma_2) &= a + 2b - c, \\ (D \cdot \Gamma_3) &= (\Gamma \cdot \Gamma_5) = -b + 2c, & (D \cdot \Gamma_4) &= -2a + b + c + d. \end{aligned}$$

So Lemma 5.1 implies the assertion. \square

REMARK 5.4. According to [19], the vector space $\text{Pic}_{\mathbb{Q}}(\overline{S}_2^+) = \text{Pic}_{\mathbb{Q}}(\overline{M}_{0,[3,3]})$ is of dimension 3. Hence there is a non-trivial relation

$$aA_0 + bB_0 + cA_1 + dB_1 = 0,$$

between the classes A_i and B_i . Intersecting with the curves Γ_i , we get the system of equations

$$3a = d, \quad a + 2b = c, \quad 2c = b, \quad b + c + d = 2a.$$

This gives the relation

$$3A_0 - 2B_0 - A_1 + 9B_1 = 0 \tag{5.5}$$

in $\text{Pic}_{\mathbb{Q}}(\overline{S}_2^+)$. Note that this is different from the relation proved in [19, Lemma 21(ii)].

Since $\text{Pic}_{\mathbb{Q}}(\overline{S}_2^+)$ is of dimension 3, (5.5) implies that $\{A_0, B_0, B_1\}$ is a basis of $\text{Pic}_{\mathbb{Q}}(\overline{S}_2^+)$. In terms of this basis, Proposition 5.3 can be rephrased as the following corollary.

COROLLARY 5.5. A divisor $D = aA_0 + bB_0 + cB_1$ of \overline{S}_2^+ is nef (respectively, ample) if and only if

$$b \leq 0 \quad \text{and} \quad \max(\frac{1}{3}c, -2b) \leq a \leq \frac{1}{2}(b + c).$$

6. Proof of part (1) of Theorem 1.2

As in Section 2, let \overline{S}_2^+ and \overline{M}_2 denote the corresponding moduli stacks of \overline{S}_2^+ and \overline{M}_2 . Consider the following commutating diagram:

$$\begin{array}{ccc} \overline{S}_2^+ & \xrightarrow{\tilde{q}} & \overline{S}_2^+ \\ \tilde{\pi} \downarrow & & \downarrow \pi \\ \overline{M}_2 & \xrightarrow{q} & \overline{M}_2 \end{array}$$

where π (respectively, $\tilde{\pi}$) is the forgetful map $[(C, \kappa)] \mapsto [C]$ and \tilde{q} and q are the natural maps. The first aim is to give a formula for the class $K_{\overline{S}_2^+}$ in terms of divisors in \overline{S}_2^+ .

Let $E \subset M_2$ the locus of the bielliptic curves. Any bielliptic curve $C \in E$ has an equation of the form

$$y^2 = (x^2 - x_1^2)(x^2 - x_2^2)(x^2 - x_3^2). \tag{6.1}$$

Thus the Weierstrass points of C are $\pm x_1, \pm x_2, \pm x_3$. Let ρ denote the bielliptic involution in C .

LEMMA 6.1. The preimage of the bielliptic locus E in \overline{S}_2^+ decomposes as

$$\pi^*(E) = \tilde{E} \cup \tilde{E}',$$

where \tilde{E} represents the spin curves (C, κ) with C admitting a bielliptic involution ρ such that $\rho^*(\kappa) = \kappa$. Moreover, the map $\pi : \tilde{E} \mapsto E$ is finite of degree 4. In particular, \tilde{E} is the bielliptic locus of \overline{S}_2^+ .

Proof. Given a bielliptic curve $[C] \in E$ with equation (6.1) and Weierstrass points $\pm x_1, \pm x_2, \pm x_3$, it suffices to check that there are exactly four even theta characteristics on C which are invariant under the action of ρ .

We set $x'_i := -x_i$ for $i = 1, 2, 3$. The involution ρ acts on the Weierstrass points by $x_i \mapsto x'_i$. The even theta characteristics on C are of the form

$$p_1 + p_2 - p_3 \sim p_4 + p_5 - p_6,$$

where $\{p_1, \dots, p_6\}$ is the set of Weierstrass points. Using this, equivalence is easy to verify that exactly the following theta characteristics are fixed by ρ :

$$x_1 + x_2 - x_3, \quad x_1 + x_2 - x'_3, \quad x_1 + x'_2 - x_3, \quad x'_1 + x_2 - x_3.$$

Thus the action of ρ decomposes π^*E into to divisors: \tilde{E} , where ρ fixes the theta characteristics and \tilde{E}' , where ρ permutes non-trivially the remaining six theta characteristics, which are

$$x_i + x'_i - x_j, \quad 1 \leq i, j \leq 3, \quad i \neq j. \quad \square$$

Recall that $\overline{M}_2 \setminus M_2 = \Delta_0 \cup \Delta_1$, where Δ_0 (respectively, Δ_1) is the closure of the locus of irreducible nodal curves (respectively, of the reducible curves) of genus 2.

LEMMA 6.2. *The forgetful map $\pi : \overline{\mathcal{S}}_2^+ \rightarrow \overline{\mathcal{M}}_2$ is simply ramified on B_0 and unramified everywhere else in codimension 1.*

Proof. Clearly, π is étale of degree 10 over the smooth locus of \mathcal{M}_2 . We have $\pi^{-1}(\Delta_0) = A_0 \cup B_0$. From the construction of the divisor A_0 , we obtain that $\deg(\pi|_{A_0}) = 4$ (since an elliptic curve admits exactly four theta characteristics). Over a semistable curve $\tilde{C} \cup R$, with $g(\tilde{C}) = 1$ and R a rational component, there are three even theta characteristics (since an elliptic curve admits exactly three even theta characteristics). From $\deg(\pi|_{B_0}) = 6$, we deduce that π is simply ramified in B_0 . One verifies that π is étale over Δ_1 since $\pi^{-1}(\Delta_1) = A_1 \cup B_1$, $\deg(\pi|_{A_1}) = 9$ and π maps B_1 isomorphically onto Δ_1 . □

We use the previous lemmas to compute the class of $K_{\overline{\mathcal{S}}_2^+}$.

LEMMA 6.3.

$$K_{\overline{\mathcal{S}}_2^+} = \tilde{q}^*(K_{\overline{\mathcal{S}}_2^+} + \frac{1}{2}(A_1 + B_1) + \frac{1}{2}\tilde{E}),$$

where \tilde{E} denotes the spin bielliptic locus of $\overline{\mathcal{S}}_2^+$.

Proof. The map \tilde{q} is ramified along the locus of points in $\overline{\mathcal{S}}_2^+$ which admits an automorphism group bigger than $\langle \iota \rangle$, with ι the hyperelliptic involution. It suffices to compute the codimension-1 components of this locus. In [14], the locus of such curves have been computed for $\overline{\mathcal{M}}_2$. It consists of the locus of bielliptic curves plus the boundary divisor Δ_1 . One immediately checks that the automorphism group of a general spin curve in $\pi^{-1}(\Delta_1) = A_1 \cup B_1$ is abelian of order 4. This together with the Lemma 6.1 proves the lemma. □

From the proof, we also get that the boundary class δ of the stack $\overline{\mathcal{S}}_2^+$ is given by

$$\delta = \tilde{q}^*(A_0 + B_0 + \frac{1}{2}A_1 + \frac{1}{2}B_1). \tag{6.2}$$

So we get as an immediate consequence.

COROLLARY 6.4 (Ramification formula for \tilde{q}). *For every $\epsilon \in \mathbb{Q}$,*

$$K_{\overline{\mathcal{S}}_2^+} + \epsilon\delta = \tilde{q}^* \left(K_{\overline{\mathcal{S}}_2^+} + \epsilon(A_0 + B_0) + \frac{\epsilon + 1}{2}(A_1 + B_1) + \frac{1}{2}\tilde{E} \right).$$

PROPOSITION 6.5. *The divisor $K_{\overline{\mathcal{S}}_2^+} + \epsilon\delta$ is nef (respectively, ample) if and only if $\epsilon \geq \frac{57}{25}$ (respectively, the inequality is strict).*

Proof. From [14, Section 4.2 and Proposition 3.6], we have

$$K_{\overline{\mathcal{M}}_2} \equiv -\frac{11}{5}\Delta_0 - \frac{16}{5}\Delta_1, \quad E \equiv 3\Delta_0 + 12\Delta_1.$$

Since by Lemma 6.2, we have $\pi^*\Delta_0 = A_0 + 2B_0$ and $\pi^*\Delta_1 = A_1 + B_1$, we get

$$\begin{aligned} K_{\overline{\mathcal{S}}_2^+} &= \pi^* \left(-\frac{11}{5}\Delta_0 - \frac{16}{5}\Delta_1 \right) + B_0 \\ &= -\frac{11}{5}A_0 - \frac{17}{5}B_0 - \frac{16}{5}A_1 - \frac{16}{5}B_1. \end{aligned}$$

Moreover, by Lemma 6.1 and the fact that π is étale of degree 10 over M_2 , we get

$$\begin{aligned} \tilde{E} &= \frac{4}{10}\pi^*(E) \\ &= \frac{4}{10}(\pi^*(3\Delta_0 + 12\Delta_1)) = \frac{6}{5}A_0 + \frac{12}{5}B_0 + \frac{24}{5}A_1 + \frac{24}{5}B_1. \end{aligned}$$

So Corollary 6.4 implies $K_{\overline{S}_2^+} + \epsilon\delta = \tilde{q}^*F$ with

$$F = \left(\epsilon - \frac{8}{5}\right)A_0 + \left(\epsilon - \frac{11}{5}\right)B_0 + \left(\frac{\epsilon}{2} - \frac{3}{10}\right)A_1 + \left(\frac{\epsilon}{2} - \frac{3}{10}\right)B_1. \tag{6.3}$$

The inequalities of Proposition 5.3 applied to F give the following conditions for the class F to be ample:

$$\epsilon > \frac{9}{5}, \quad \epsilon > \frac{57}{25}$$

(the last two inequalities do not impose conditions on ϵ). □

Let us recall the definition of a log canonical model. Let X be a normal projective variety and $D = \sum_{i=1}^n a_i D_i$ be a \mathbb{Q} -divisor such that $K_X + D$ is \mathbb{Q} -Cartier and $0 \leq a_i \leq 1$.

DEFINITION 6.6. The pair (X, D) is a *strict log canonical model* if $K_X + D$ is ample, (X, D) has log canonical singularities, and $X \setminus \bigcup_i D_i$ has canonical singularities.

We will also need the following proposition, whose proof we refer the reader to [17, 20.2, 20.3]).

PROPOSITION 6.7. *Let Y be a smooth variety and $f : Y \rightarrow X$ be a finite dominant morphism to a normal variety X . Let $D = \sum_{i=1}^n a_i D_i$, $0 \leq a_i \leq 1$ be a \mathbb{Q} -divisor containing all the divisorial components of the branch locus of f . Let \overline{D} be a \mathbb{Q} -divisor on Y such that $\text{supp}(f^{-1}(D)) = \text{supp}(\overline{D})$ and $f^*(K_X + D) = K_Y + \overline{D}$. Then (X, D) has log canonical singularities along D if and only if (Y, \overline{D}) has log canonical singularities along \overline{D} .*

Proof of Theorem 1.2(1). Proposition 6.5 shows the required ampleness of the divisor F . In order to verify the singularity conditions of \overline{S}_2^+ , we will use the results in [14, Theorem 4.10]. Consider the finite map $\pi : \overline{S}_2^+ \rightarrow \overline{M}_2$. We have that \overline{M}_2 has canonical singularities away from Δ_0 and Δ_1 (see [14]). Since $\pi|_{\overline{S}_2^+}$ is étale, it follows that \overline{S}_2^+ has canonical singularities away from $\pi^{-1}(\Delta_0 \cup \Delta_1) = A_0 \cup B_0 \cup A_1 \cup B_1$. In particular, it has canonical singularities away from the boundary divisors A_0, B_0, A_1, B_1 and \tilde{E} .

Since \overline{M}_2 has log canonical singularities along Δ_0, Δ_1 and E (see [14]), by Proposition 6.7, \overline{S}_2^+ has log canonical singularities along $\pi^{-1}(\Delta_0 \cup \Delta_1 \cup E)$, and hence also along $A_0 \cup B_0 \cup A_1 \cup B_1 \cup \tilde{E}$, since \overline{M}_2 is smooth at E and π is étale over E . □

7. The invariant-theoretical compactification of S_2^+

Let C be a smooth projective curve of genus 2 over k . Let \mathbb{F}_2^4 denote the four-dimensional vector space over \mathbb{F}_2 equipped with a fixed symplectic form. The group of 2-division points $JC[2]$ of the Jacobian of C is isomorphic to \mathbb{F}_2^4 and admits a canonical symplectic form, the Weil form. A *curve of genus 2 with a level-2 structure* consists of a pair (C, φ) with a curve C of genus 2 and a *level-2 structure* φ , that is, a symplectic isomorphism $\varphi : \mathbb{F}_2^4 \rightarrow JC[2]$. Let $M_2(2)$ denote the coarse moduli space of such pairs. It can be constructed as follows:

The hyperelliptic covering $C \rightarrow \mathbb{P}^1$ is ramified exactly in the six Weierstrass points p_1, \dots, p_6 with images $x_1, \dots, x_6 \in \mathbb{P}^1$. It is well known (see [11]) that a level-2 structure on C is equivalent to an order of the set of Weierstrass points of C . Denote by

$$\Delta_6 = \{(x_1, \dots, x_6) \in (\mathbb{P}^1)^6 \mid x_i = x_j \text{ for some } i \neq j\}$$

the diagonal of the sixfold cartesian product $(\mathbb{P}^1)^6$ of \mathbb{P}^1 . Then

$$M_2(2) \simeq [(\mathbb{P}^1)^6 \setminus \Delta_6] // \mathrm{SL}_2(k),$$

where $\mathrm{SL}_2(k)$ acts in the usual way on \mathbb{P}^1 and diagonally on $(\mathbb{P}^1)^6$. The forgetful map

$$M_2(2) \rightarrow M_2, \quad (C, \varphi) \mapsto C,$$

onto the coarse moduli space M_2 of smooth curves of genus 2 corresponds to the quotient map

$$[(\mathbb{P}^1)^6 \setminus \Delta_6] // \mathrm{SL}_2(k) \rightarrow \{[(\mathbb{P}^1)^6 \setminus \Delta_6] // \mathrm{SL}_2(k)\} / \mathfrak{S}_6,$$

where the action of the symmetric group \mathfrak{S}_6 is induced by its permutation action on $(\mathbb{P}^1)^6$.

Recall from [22] that an even theta characteristic on C is the line bundle given by a divisor $p_{i_1} + p_{i_2} - p_{i_3}$, where the p_{i_j} are different Weierstrass points and $p_{j_1} + p_{j_2} - p_{j_3}$ defines the same theta characteristic if and only if either $\{j_1, j_2, j_3\} = \{i_1, i_2, i_3\}$ or $\{i_1, i_2, i_3, j_1, j_2, j_3\} = \{1, \dots, 6\}$. This implies that the even theta characteristics of C are in a natural bijective correspondence with the 3-element subsets of the set $\{1, \dots, 6\}$ modulo the involution $\{i_1, i_2, i_3\} \mapsto \{1, \dots, 6\} \setminus \{i_1, i_2, i_3\}$. In order to construct the coarse, moduli space S_2^+ of even spin curves of genus 2, consider again the subgroup

$$G := (\mathfrak{S}_3 \times \mathfrak{S}_3) \rtimes \langle \tau \rangle$$

of \mathfrak{S}_6 as defined at the beginning of Section 5.

Clearly, the stabilizer of an even theta characteristic given by $\{i_1, i_2, i_3\}$ is conjugate to G . Hence we obtain an isomorphism

$$S_2^+ \xrightarrow{\simeq} M_2(2)/G = \{[(\mathbb{P}^1)^6 \setminus \Delta_6] // \mathrm{SL}_2(k)\} / G. \tag{7.1}$$

In [11, p. 17], it is shown that a natural compactification of $M_2(2)$, the invariant-theoretical compactification, which we denote by $\overline{M_2(2)}^{\mathrm{inv}}$, is isomorphic to the Segre cubic threefold. In $\mathbb{P}^5 = \mathbb{P}^5(t_0, \dots, t_5)$, the Segre cubic is given by the equations

$$s_1 := \sum_{i=0}^5 t_i = 0 \quad \text{and} \quad s_3 := \sum_{i=0}^5 t_i^3 = 0, \tag{7.2}$$

where \mathfrak{S}_6 acts by permuting the coordinates. In other words,

$$\overline{M_2(2)}^{\mathrm{inv}} = \mathrm{Proj}(k[t_0, \dots, t_5] / (s_1, s_3)). \tag{7.3}$$

Together with (7.1), this implies the existence of a natural compactification $\overline{S_2^+}^{\mathrm{inv}}$, the invariant-theoretical compactification of S_2^+ , given by

$$\overline{S_2^+}^{\mathrm{inv}} = \mathrm{Proj}((k[t_0, \dots, t_5] / (s_1, s_3))^G). \tag{7.4}$$

Here $(k[t_1, \dots, t_5] / (s_1, s_3))^G$ denotes the ring of invariants in $\mathbb{C}[t_0, \dots, t_5] / (s_1, s_3)$, where the first \mathfrak{S}_3 acts by permuting t_0, t_1, t_2 , the second \mathfrak{S}_3 by permuting t_3, t_4, t_5 , and τ by exchanging t_i and t_{i+3} for $i = 0, 1, 2$.

The canonical map $M_2(2) \rightarrow M_2$ factorizes as

$$\begin{array}{ccc} M_2(2) & \xrightarrow{f} & S_2^+ \\ & \searrow & \swarrow g \\ & M_2 & \end{array}$$

where g is étale of degree 10. If we denote by $s_k = s_k(t_0, \dots, t_5) := \sum_{j=0}^5 t_j^k$ for $k = 1, \dots, 6$, then the corresponding rings of invariants are

$$k[t_0, \dots, t_5]/(s_1, s_3) \supset (k[t_0, \dots, t_5]/(s_1, s_3))^G \supset (k[t_0, \dots, t_5]/(s_1, s_3))^{\mathfrak{S}_6} = k[s_2, s_4, s_5, s_6],$$

where the last equality holds since \mathfrak{S}_6 acts by permuting the t_i . Taking the Proj of these rings gives the commutative diagram

$$\begin{array}{ccc} \overline{M_2(2)}^{\text{inv}} & \xrightarrow{\bar{f}} & \overline{S_2^+}^{\text{inv}} \\ & \searrow & \swarrow \bar{g} \\ & \overline{M_2}^{\text{inv}} & \end{array} \tag{7.5}$$

which compactifies the above diagram by (7.3) and (7.4). Note that, since the ring extensions are finite, the maps \bar{f} and \bar{g} are everywhere defined and finite. According to the following remark, $\overline{M_2}^{\text{inv}}$ is the classical invariant-theoretical compactification of M_2 .

REMARK 7.1. With the above coordinates, we have

$$\overline{M_2}^{\text{inv}} = \text{Proj}(k[s_2, s_4, s_5, s_6]) = \mathbb{P}(2, 4, 5, 6).$$

On the other hand, in terms of the invariant of binary sextics (see [15] or [14]),

$$\overline{M_2}^{\text{inv}} = \text{Proj}(k[A, B, C, D]) = \mathbb{P}(1, 2, 3, 5),$$

where A, B, C, D are the classical invariants degree 2, 4, 6, 10, respectively. By [9, Proposition of Delorme], there is a natural isomorphism

$$\mathbb{P}(2, 4, 5, 6) \simeq \mathbb{P}(1, 2, 3, 5).$$

REMARK 7.2. The ring of invariants $(k[t_0, \dots, t_5]/(s_1, s_3))^G$ was explicitly computed by Clebsch [7, Section 61]. For a modern version, see [18, Section 4]. We do not need the explicit form of the ring.

REMARK 7.3. The map \bar{g} is of degree

$$\text{deg } \bar{g} = \frac{|S_6|}{|G|} = \frac{720}{72} = 10,$$

which coincides with the fact that every smooth curve of genus 2 admits exactly ten even theta characteristics.

In [11], the variety $\overline{M_2(2)}^{\text{inv}}$ is interpreted as the moduli space of semistable ordered sets of six points in \mathbb{P}^1 . As such, the strictly semistable sets are given by the ten singular points of $\overline{M_2(2)}^{\text{inv}}$. Here we consider $\overline{M_2(2)}^{\text{inv}}$ as the moduli space of semistable curves of genus 2 with

level-2 structure. We want to determine the image of the boundary divisor $\overline{M_2(2)}^{\text{inv}} \setminus M_2(2)$ in $\overline{S_2}^{+\text{inv}}$.

Let $D_2(2), D_2^+$ and D_2 the boundary divisors of $\overline{M_2(2)}^{\text{inv}}, \overline{S_2}^{+\text{inv}}$ and $\overline{M_2}^{\text{inv}}$, that is, $D_2(2)$ is the closed subscheme $\overline{M_2(2)}^{\text{inv}} \setminus M_2(2)$ of $\overline{M_2(2)}^{\text{inv}}$, etc. The above two diagrams induce the following diagram of finite surjective maps:

$$\begin{array}{ccc} D_2(2) & \xrightarrow{\bar{f}} & D_2^+ \\ & \searrow & \swarrow \bar{g} \\ & D_2 & \end{array}$$

We clearly have

$$D_2(2) = \Delta_6 = \bigcup_{1 \leq i < j \leq 6} D_{ij} \quad \text{with } D_{ij} = \{(x_1, \dots, x_6) \in (\mathbb{P}^1)^6 \mid x_i = x_j\}.$$

Denoting

$$A_0^{\text{inv}} := \bar{f}(D_{12}) \quad \text{and} \quad B_0^{\text{inv}} := \bar{f}(D_{14}),$$

we have the following proposition.

PROPOSITION 7.4. *The divisor D_2^+ consists of the two irreducible components*

$$D_2^+ = A_0^{\text{inv}} \cup B_0^{\text{inv}},$$

where A_0^{inv} (respectively, B_0^{inv}) is the closure of points parametrizing irreducible spin curves with one node (respectively, spin curves with one exceptional component and irreducible stable reduction).

Proof. The group G acts on the set of components of $D_2(2)$ with two orbits represented by D_{12} and D_{14} , which gives the first assertion. The geometric interpretation of the components follows from [19] as in (5.1) and (5.2). This explains also the notation. \square

According to [10, Remark after Theorem 9.4.10, p. 526], the boundary divisor $D_2(2)$ of $\overline{M_2(2)}^{\text{inv}} = \text{Proj}(k[t_0, \dots, t_5]/(s_1, s_3))$ is given by the fifteen planes in the Segre cubic with equations

$$t_i + t_j = t_k + t_l = t_m + t_n = 0,$$

where $\{i, j, k, l, m, n\} = \{1, \dots, 6\}$. The group G acts on them with two orbits represented by the planes Π_1 with equations $t_0 + t_3 = t_1 + t_4 = t_2 + t_5 = 0$ and Π_2 with equations $t_0 + t_1 = t_2 + t_3 = t_4 + t_5 = 0$. The orbit of Π_1 consists of six planes, whereas the orbit of Π_2 consists of nine planes. Comparing with (5.1) and (5.2), this implies

$$f^*(A_0^{\text{inv}}) = \text{Orb}_G(\Pi_1) \quad \text{and} \quad f^*(B_0^{\text{inv}}) = \text{Orb}_G(\Pi_2),$$

or equivalently

$$A_0^{\text{inv}} = f(\Pi_1) \quad \text{and} \quad B_0^{\text{inv}} = f(\Pi_2),$$

since the natural map $\overline{M_2(2)} \rightarrow \overline{M_2(2)}^{\text{inv}}$ is G -equivariant.

The Segre cubic threefold $\overline{M_2(2)}^{\text{inv}}$ contains exactly ten singular points, that is, nodes (see [11, Example 2, p. 31]). Their coordinates are $(\pm 1, \dots, \pm 1)$, where exactly half of them are positive. Here the equation is taken in the \mathfrak{S}_6 -invariant form (7.2). Denoting

$$a_1^{\text{inv}} := \bar{f}(1, 1, -1, 1, -1, -1) \quad \text{and} \quad b_1^{\text{inv}} := \bar{f}(1, 1, 1, -1, -1, -1),$$

we have the following proposition.

PROPOSITION 7.5. *The point a_1^{inv} (respectively, b_1^{inv}) in $\overline{S}_2^{+\text{inv}}$ represents all stable spin curves with 2 smooth genus-1 components connected by one exceptional component with even (respectively, odd) theta characteristics on the elliptic curves.*

Proof. The group G acts on ten singular points of $\overline{M}_2(2)^{\text{inv}}$ with orbits represented by $(1, 1, -1, 1, -1, -1)$ and $(1, 1, 1, -1, -1, -1)$. The orbit of $(1, 1, -1, 1, -1, -1)$ consists of nine singular points, whereas $(1, 1, 1, -1, -1, -1)$ is a fixed point under the action. Comparing with (5.3) and (5.4) and the definition of A_1 and B_1 gives the assertion. \square

8. The map $\overline{S}_2^+ \rightarrow \overline{S}_2^{+\text{inv}}$

We have constructed two compactifications of the moduli space S_2^+ of smooth even spin curves of genus 2, namely the moduli space \overline{S}_2^+ of generalized even spin curves of genus 2 and the invariant-theoretical compactification $\overline{S}_2^{+\text{inv}}$. They fit into the following diagram:

$$\begin{array}{ccc} \overline{S}_2^+ & & \overline{S}_2^{+\text{inv}} \\ \pi \downarrow & & \downarrow \bar{g} \\ \overline{M}_2 & \xrightarrow{f_2} & \overline{M}_2^{\text{inv}} \end{array}$$

where π denotes the forgetful map, \bar{g} the map of diagram (7.5) and f_2 is the canonical holomorphic map constructed in [4, Theorem 1.1].

PROPOSITION 8.1. *There is a canonical holomorphic map $\bar{f}_2 : \overline{S}_2^+ \rightarrow \overline{S}_2^{+\text{inv}}$ making the following diagram commutative:*

$$\begin{array}{ccc} \overline{S}_2^+ & \xrightarrow{\bar{f}_2} & \overline{S}_2^{+\text{inv}} \\ \pi \downarrow & & \downarrow \bar{g} \\ \overline{M}_2 & \xrightarrow{f_2} & \overline{M}_2^{\text{inv}} \end{array} \tag{8.1}$$

Moreover, \bar{f}_2 contracts the divisor A_1 to the point a_1^{inv} , the divisor B_1 to the point b_1^{inv} and is biholomorphic on $\overline{S}_2^+ \setminus (A_1 \cup B_1)$.

Proof. Both spaces \overline{S}_2^+ and $\overline{S}_2^{+\text{inv}}$ are compactifications of S_2^+ , which gives a canonical birational map \bar{f}_2 making (8.1) commutative. The boundary of \overline{S}_2^+ consists of the divisors A_0, B_0, A_1, B_1 and the boundary of $\overline{S}_2^{+\text{inv}}$ consists of the divisors $A_0^{\text{inv}}, B_0^{\text{inv}}$ and the points $a_1^{\text{inv}}, b_1^{\text{inv}}$. The spaces $A_0 \setminus (A_1 \cup B_1)$ and $A_0^{\text{inv}} \setminus (a_1^{\text{inv}} \cup b_1^{\text{inv}})$ (respectively, $B_0 \setminus (A_1 \cup B_1)$ and $B_0^{\text{inv}} \setminus (a_1^{\text{inv}} \cup b_1^{\text{inv}})$) parametrize the same objects. Hence the map \bar{f}_2 extends to them.

On the other hand, the divisors A_1 and B_1 lie over the boundary divisor Δ_1 of \overline{M}_2 , whereas the points a_1^{inv} and b_1^{inv} lie over the boundary point p_{ss} (see [3, Corollary 5.3]) of $\overline{M}_2^{\text{inv}}$. Since f_2 contracts Δ_1 to p_{ss} , the last assertion follows from Proposition 7.5. \square

9. Proof of part (2) and (3) of Theorem 1.2

For the sake of abbreviation, let $Y := \overline{S}_2^{+inv}$ and $X := \overline{M}_2^{inv}$ be the coarse moduli spaces, and we denote by $\tilde{q} : \mathcal{Y} \rightarrow Y$ and $q : \mathcal{X} \rightarrow X$ the corresponding moduli stacks. We have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\tilde{q}} & Y \\ \tilde{g} \downarrow & & \downarrow \bar{g} \\ \mathcal{X} & \xrightarrow{q} & X \end{array}$$

where \bar{g} denotes the forgetful map of diagram 8.1 and \tilde{g} the corresponding map on the stack level. Recall from Proposition 7.4 that the boundary of Y is $D_2^+ = A_0^{inv} \cup B_0^{inv}$. Similarly, the boundary of \mathcal{Y} is $\mathcal{D}_2^+ = \alpha_0^{inv} + \beta_0^{inv}$.

Since $\text{Pic}_{\mathbb{Q}}(\overline{S}_2^+)$ is generated by A_0, B_0, A_1, B_1 with one relation (5.5), we deduce by means of Proposition 8.1 that $\text{Pic}(Y)$ is generated by the classes A_0^{inv} and B_0^{inv} with the relation

$$3A_0^{inv} = 2B_0^{inv}. \tag{9.1}$$

It follows from Lemma 6.2 and Proposition 8.1 that the map $\bar{g} : Y \rightarrow X$ is simply ramified in B_0^{inv} and unramified elsewhere in codimension 1. It is well known that $\text{Pic}_{\mathbb{Q}}(X)$ is generated by the class of the boundary Δ_0 . Hence

$$\bar{g}^*(\Delta_0) = A_0^{inv} + 2B_0^{inv}.$$

Further we deduce from Lemma 6.1 and the fact that the bielliptic locus of X is given by the class $3\Delta_0$ (see [14]), that the bielliptic locus of Y is given by the class

$$\tilde{E} = \frac{2}{5} \cdot 3\bar{g}^*(\Delta_0) = \frac{24}{5}A_0^{inv}.$$

As for Corollary 6.4, we obtain the following formula for $\tilde{q} : \mathcal{Y} \rightarrow Y$. For every $\epsilon \in \mathbb{Q}$

$$K_{\mathcal{Y}} + \epsilon\mathcal{D}_2^+ = \tilde{q}^*(\pi^*K_X + B_0 + \frac{1}{2}\tilde{E} + \epsilon(D_2^+))$$

holds. Using the relation (9.1) and noting that according to [14], we have

$$K_X = -\frac{11}{5}\Delta_0,$$

we get the following ramification formula for $\tilde{q} : \mathcal{Y} \rightarrow Y$.

PROPOSITION 9.1. For every $\epsilon \in \mathbb{Q}$, we have

$$K_{\mathcal{Y}} + \epsilon(\mathcal{D}_2^+) = \tilde{q}^*((-\frac{49}{10} + \frac{5}{2}\epsilon)A_0^{inv}).$$

Proof of parts (2) and (3) of Theorem 1.2. Proposition 9.1 implies that the class $(-\frac{49}{10} + \frac{5}{2}\epsilon)A_0^{inv}$ is nef (respectively, ample) if and only if $\epsilon \geq \frac{49}{25}$ (respectively, $\epsilon > \frac{49}{25}$).

It remains to check the singularity conditions of Y . Note first that the boundary of Y is $A_0^{inv} \cup B_0^{inv}$. Moreover, the divisors A_0 and B_0 intersect non-trivially in \overline{S}_2^+ , Proposition 8.1 implies that the contraction points a_1^{inv} and b_1^{inv} are contained in $A_0^{inv} \cup B_0^{inv}$. Hence the proof is the same as in the proof of part (1) of Theorem 1.2 in Section 6. \square

10. Proof of part (2) of Theorem 1.3

Recall that α is a canonical isomorphism $\alpha : \mathfrak{S}_3\widehat{M}_2 \rightarrow S_2^+$ and $\mathfrak{S}_3\widetilde{M}_2$ the partial compactification $\mathfrak{S}_3\widetilde{M}_2 = \mathfrak{S}_3\widehat{M}_2 \sqcup S$ of [21], with S the closed subscheme of $\mathfrak{S}_3\widehat{M}_2$ consisting of reducible

\mathfrak{S}_3 -coverings. In Section 4, we proved that the construction of the map α does not extend to a holomorphic map $\mathfrak{S}_3\widetilde{M}_2 \rightarrow \overline{S}_2^+$. However, we show the following proposition.

PROPOSITION 10.1. *The map $\alpha : \mathfrak{S}_3\widehat{M}_2 \rightarrow S_2^+$ extends to a holomorphic map $\bar{\alpha} : \mathfrak{S}_3\widetilde{M}_2 \rightarrow \overline{S}_2^{+inv}$ which contracts the divisor S to the point a_1^{inv} in S_2^{+inv} .*

Proof. According to Lemma 4.1, the map α extends to a birational map $\tilde{\alpha} : \mathfrak{S}_3\widetilde{M}_2 \rightarrow S_2^+$ induced by the construction in Section 3 defined on $\mathfrak{S}_3\widehat{M}_2 \cup U$, with $U \subset S$ a non-empty open set. Since the image of the map α degenerates to a product of elliptic curves when a covering in $\mathfrak{S}_3\widehat{M}_2$ degenerates to a spin curve in S , that is, with underlying reducible curve, the image $\tilde{\alpha}(U)$ is contained in $A_1 \cup B_1$.

Consider $(X_t, \kappa_t) \in S_2^+$ a family of even spin smooth curves in the image of α degenerating to an admissible covering $X_0 = C_1 \cup C_2$ with C_1, C_2 elliptic curves and $C_1 \cap C_2 = \{p\}$ in the following way. If $\kappa_t = \mathcal{O}_{X_t}(\omega_{1,t} - \omega_{2,t} + \omega_{3,t})$ for some Weierstrass points $\omega_{i,t}$, $i = 1, 2, 3$, then each element in the family comes with a 6:1 map $\psi_t : X_t \rightarrow \mathbb{P}^1$ given by the pencil $\langle 2\omega_{1,t} + 2\omega_{2,t} + 2\omega_{3,t}, 2\omega_{4,t} + 2\omega_{5,t} + 2\omega_{6,t} \rangle \subset |3K_{X_t}|$ (see Section 3). The Weierstrass points $\omega_{i,t}$ specialize to points $\omega_{i,0}$ lying on the component C_1 for $i = 1, 2, 3$ and on C_2 for $i = 4, 5, 6$. One checks that such admissible covering $X_0 \rightarrow \mathbb{P}^1 \cup \mathbb{P}^1$ must be totally ramified at the node p , which is the neutral element for both elliptic curves and the 6:1 map $C_1 \rightarrow \mathbb{P}^1$ (respectively, $C_2 \rightarrow \mathbb{P}^1$) is determined by the pencil $\langle 2\omega_{1,0} + 2\omega_{2,0} + 2\omega_{3,0}, 6p \rangle$ (respectively, $\langle 2\omega_{4,0} + 2\omega_{5,0} + 2\omega_{6,0}, 6p \rangle$). Then the theta characteristic κ_t specializes to $\kappa_0 = (\kappa_{C_1}, \kappa_{C_2})$, where $\kappa_{C_1} = \mathcal{O}_{C_1}(\omega_{1,0} - \omega_{2,0} + \omega_{3,0} - p)$ and $\kappa_{C_2} = \mathcal{O}_{C_2}(\omega_{4,0} - \omega_{5,0} + \omega_{6,0} - p)$. Clearly, these 2-torsion points are non-trivial, so $(X_0, \kappa_0) \in A_1$. We conclude that $\tilde{\alpha}(U) \subset A_1$.

Since A_1 is contracted to the point a_1^{inv} in \overline{S}_2^{+inv} under the map \bar{f}_2 , there exists a map $\bar{\alpha}$ well defined on $\mathfrak{S}_3\widetilde{M}_2$ making commutative the following diagram:

$$\begin{array}{ccc}
 \mathfrak{S}_3\widetilde{M}_2 & \xrightarrow{\tilde{\alpha}} & \overline{S}_2^+ \\
 & \searrow \bar{\alpha} & \downarrow \bar{f}_2 \\
 & & \overline{S}_2^{+inv}
 \end{array}$$

and such that $\bar{\alpha}(S) = \{a_1^{inv}\}$. □

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