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Further results on modified harmonic functions in three dimensions

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The Weinstein equation $t\Delta u+k\frac{\partial u}{\partial t}=0$, with $k\in\mathbb{Z}$, considered in $\mathbb{R}^3=\{(x,y,t)\}$, is a modification of the classical Laplace equation $\Delta u = 0$. Its solutions are called k-modified harmonic functions. Whereas for positive integers k the Weinstein equation is relatively well understood, little is known if the parameter k is negative.
The main result of this article is the statement that in case the negative integers are even, i.e., $k=-2\ell$, $\ell\in\mathbb{N}$, we still have a Fischer-type decomposition. For $k=0$, the classical harmonic functions, this decomposition is well known. But also in case $k\in\mathbb{N}$, a Fischer-type decomposition holds true, a Fischer-type decomposition holds true. Surprisingly in case $k=-3$, $k=-5$, or $k=-7$ and probably in all higher negative odd cases, the decomposition doesn't hold.
In case $k=-1$, we give a complete description of the vector space $\mathcal{H}_n^k(\mathbb{R}^3)$ of homogeneous k-modified harmonic polynomials of degree n in $\mathbb{R}^3$. Such a result is also at hand in case $k\in\mathbb{N}$. Finally, in case $k=0$ of the classical harmonic functions, we give a description of the vector space $\mathcal{H}_n(\mathbb{R}^3)=\mathcal{H}_n^0(\mathbb{R}^3)$.

KEYWORDS
generalized axially symmetric potentials, modified spherical harmonics, spherical harmonics

MSC CLASSIFICATION
30G35; 33A45

1 | INTRODUCTION

The aim of this article is to study the solutions of the Weinstein equation

$$t\Delta u+k\frac{\partial u}{\partial t}=0,\ k\in\mathbb{Z}, \tag{1}$$

in $\mathbb{R}^3=\{(x,y,t)\}$, in case k is a negative integer, $k\in-\mathbb{N}$.

Hereby, as usual,

$$\Delta := \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial t^2}$$

denotes the Laplace operator.

The solutions of Equation (1) are often called generalized axially symmetric potentials for the following reason:

The function $u = u(x, y, t)$, defined in a subdomain of the half space $\mathbb{R}^3_+ = \{(x, y, t), t > 0\}$, is a solution of (1) if and only if, in case $k \in \mathbb{N}$, the function

$$w(\xi_1, \xi_2, \dots, \xi_{k+3}) := u\left(\xi_1, \xi_2, \sqrt{\xi_3^2 + \dots + \xi_{k+3}^2}\right)$$

is harmonic—in the classical sense—in the corresponding domain in $(k+3)$-dimensional space.

By some authors (see, e.g., Eriksson and Orelma[1] or their other study[2]) solutions of (1) are also called k-hyperbolic harmonic functions. But we shall not use these notations here. Instead, we introduce the following definition:

Definition 1. Solutions of the equation (1) will be called k-modified harmonic functions.

The equation (1) is the Laplace-Beltrami equation associated with the Riemannian metric

$$ds^2 = t^{2k}(dx^2 + dy^2 + dt^2),$$

since the corresponding Laplace-Beltrami operator Δ_{LB} is given by

$$\Delta_{LB} := \frac{1}{t^{2k}}\left(\Delta + \frac{k}{t}\frac{\partial}{\partial t}\right).$$

The equation (1) has already been considered by several authors. A good reference is Brelot's article.[3] We also refer to Akin and Leutwiler[4] for further investigations.

The behavior of the solutions of Equation (1) in the three cases $k \in \mathbb{N}$, $k = 0$, and $k \in -\mathbb{N}$ is completely different. Apart from the case $k = 0$, the classical harmonic functions, the case $k \in \mathbb{N}$ is relatively well understood. See previous studies.[5-10] The main differences between the cases $k \in \mathbb{N}$ and $k \in -\mathbb{N}$ are the facts that in case $k \in -\mathbb{N}$, we no longer have an appropriate scalar product on the unit half sphere nor do we have the identification principle of A. Weinstein.[11] The latter says:

If $\Omega \subset \mathbb{R}^3$ is a domain (open connected set) with $G := \Omega \cap \{(x, y, 0)\} \neq \emptyset$, every k-modified harmonic function ($k \in \mathbb{N}$) on Ω is uniquely determined by its values on the intersection $G \subset \mathbb{R}^2 = \{(x, y, 0)\}$.

We shall now start our study of the case $k \in -\mathbb{N}$ with an extension of the Fischer-type decomposition as treated in Leutwiler[5] and in previous studies.[7-10] Recall that in case $k \in \mathbb{N}$, we considered the vector space $\mathcal{P}_n(\mathbb{R}^3)$ consisting of all homogeneous polynomials $p = p(x, y, t)$ of degree n in $\mathbb{R}^3$, which are *even in the last variable* t. Besides this vector space we also looked, for $k \in \mathbb{N}$, at the vector spaces $\mathcal{H}_n^k(\mathbb{R}^3)$ of all homogeneous k-modified harmonic polynomials of degree n in $\mathbb{R}^3$. Whereas for $k \in \mathbb{N}$ the Fischer-type decomposition always exists, see Leutwiler,[9] this is no longer true for all $k \in -\mathbb{N}$. In fact, it is easy to see that it does not hold in case $k = -3$. Indeed, assuming that for some $a, b, c, d \in \mathbb{R}$

$$x^2 = axy + b(2x^2 + t^2) + c(2y^2 + t^2) + d(x^2 + y^2 + t^2) \in \mathcal{H}_2^{-3}(\mathbb{R}^3) \oplus r^2\mathcal{H}_0^{-3}(\mathbb{R}^3), \tag{2}$$

setting $t = 0$, leads to

$$x^2 = axy + 2bx^2 + 2cy^2 + d(x^2 + y^2),$$

and hence to $a = 0$, $b = \frac{1}{2}(1 - d)$ and $c = -\frac{1}{2}d$. Inserted in (2), there results the contradiction $x^2 = x^2 + \frac{1}{2}t^2$, for all $x, t \in \mathbb{R}$.

Similar reasoning, applied to x^3 in case $k = -5$, respectively x^4 in case $k = -7$, leads to the fact that in these 3 cases the Fischer-type decompostion does not hold. It is likely that for all $k = -(2\ell + 1), \ell \in \mathbb{N}$, the power $x^{\ell+1}$ yields a counterexample to the Fischer-type decompostion, but we can't verify it.

On the other hand, we can prove

Theorem 1.1. *For all $k \in \mathbb{Z}\backslash\{-3, -5, -7, -9, \dots\}$ the Fischer-type decomposition*

$$\mathcal{P}_n(\mathbb{R}^3) = \mathcal{H}_n^k(\mathbb{R}^3) \oplus r^2\mathcal{H}_{n-2}^k(\mathbb{R}^3) \oplus r^4\mathcal{H}_{n-4}^k(\mathbb{R}^3) \oplus \dots \oplus r^{2i}\mathcal{H}_{n-2i}^k(\mathbb{R}^3), \; i = \left[\frac{n}{2}\right],$$

where $r^2 = x^2 + y^2 + t^2$, exists.

Examples.

In $\mathcal{P}_2(\mathbb{R}^3)$ we have

$$\begin{aligned}
x^2 &= \tfrac{1}{k+3}[(k+2)x^2 - y^2 - t^2] + \tfrac{1}{k+3}r^2 &&\in \mathcal{H}_2^k(\mathbb{R}^3) \oplus r^2\mathcal{H}_0^k(\mathbb{R}^3)\\
y^2 &= \tfrac{1}{k+3}[-x^2 + (k+2)y^2 - t^2] + \tfrac{1}{k+3}r^2 &&\in \mathcal{H}_2^k(\mathbb{R}^3) \oplus r^2\mathcal{H}_0^k(\mathbb{R}^3)\\
t^2 &= \tfrac{1}{k+3}[-(k+1)(x^2+y^2) + 2t^2] + \tfrac{k+1}{k+3}r^2 &&\in \mathcal{H}_2^k(\mathbb{R}^3) \oplus r^2\mathcal{H}_0^k(\mathbb{R}^3)\\
xy &= xy + 0\cdot r^2 \in \mathcal{H}_2^k(\mathbb{R}^3) \oplus r^2\mathcal{H}_0^k(\mathbb{R}^3)
\end{aligned}$$

In $\mathcal{P}_3(\mathbb{R}^3)$ we have

$$\begin{aligned}
x^3 &= \tfrac{1}{k+5}[(k+2)x^3 - 3xy^2 - 3xt^2] + \tfrac{3}{k+5}xr^2 &&\in \mathcal{H}_3(\mathbb{R}^3) \oplus r^2\mathcal{H}_1(\mathbb{R}^3)\\
x^2y &= \tfrac{1}{k+5}[(k+4)x^2y - y^3 - yt^2] + \tfrac{1}{k+5}yr^2 &&\in \mathcal{H}_3(\mathbb{R}^3) \oplus r^2\mathcal{H}_1(\mathbb{R}^3)\\
xy^2 &= \tfrac{1}{k+5}[-x^3 + (k+4)xy^2 - xt^2] + \tfrac{1}{k+5}xr^2 &&\in \mathcal{H}_3(\mathbb{R}^3) \oplus r^2\mathcal{H}_1(\mathbb{R}^3)\\
y^3 &= \tfrac{1}{k+5}[-3x^2y + (k+2)y^3 - 3yt^2] + \tfrac{3}{k+5}yr^2 &&\in \mathcal{H}_3(\mathbb{R}^3) \oplus r^2\mathcal{H}_1(\mathbb{R}^3)\\
xt^2 &= \tfrac{1}{k+5}[-(k+1)(x^3+xy^2) + 4xt^2] + \tfrac{k+1}{k+5}xr^2 &&\in \mathcal{H}_3(\mathbb{R}^3) \oplus r^2\mathcal{H}_1(\mathbb{R}^3)\\
yt^2 &= \tfrac{1}{k+5}[-(k+1)(x^2y+y^3) + 4yt^2] + \tfrac{k+1}{k+5}yr^2 &&\in \mathcal{H}_3(\mathbb{R}^3) \oplus r^2\mathcal{H}_1(\mathbb{R}^3)
\end{aligned}$$

and so on.

Theorem 1.1 has already been verified for $k \in \mathbb{N}$ in Leutwiler[9] with a short proof suggested by Roman Lávička. In case $k = 0$, it even holds for the vector space of all homogeneous polynomials of degree n in $\mathbb{R}^3$, not only for $\mathcal{P}_n(\mathbb{R}^3)$. So we shall concentrate from now on $k \in \{-1\} \cup \{-2, -4, -6, \dots\}$. The proof for these cases will be an extension of the one given for $k = 1$ in Leutwiler.[5]

It will be based on the following

Lemma 1.2. *If u is a k-modified harmonic function, so are*

$$\begin{aligned}
u_1(x,y,t) &= \tfrac{\partial u}{\partial x}(x,y,t)\\
u_2(x,y,t) &= \tfrac{\partial u}{\partial y}(x,y,t)\\
u_3(x,y,t) &= (k+1)xu + (x^2 - y^2 - t^2)\tfrac{\partial u}{\partial x} + 2xy\tfrac{\partial u}{\partial y} + 2xt\tfrac{\partial u}{\partial t}\\
u_4(x,y,t) &= (k+1)yu + 2xy\tfrac{\partial u}{\partial x} + (-x^2 + y^2 - t^2)\tfrac{\partial u}{\partial y} + 2yt\tfrac{\partial u}{\partial t}
\end{aligned}$$

The proof of this lemma is based on the so-called *Kelvin transform* which reads as follows:
If u is a k-modified harmonic function, so is

$$K[u](x,y,t) = \frac{1}{r^{k+1}}u\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{t}{r^2}\right). \tag{3}$$

This fact can easily be verified by straightforward, but tedious, differentiation.

Proof of Lemma 1.2. That u_1 and u_2 are k-modified harmonic functions is obvious. Also note that in contrast $\frac{\partial u}{\partial t}$ is not necessarily a k-modified harmonic function.

In order to prove that u_3 is a k-modified harmonic function we apply the Kelvin transform. From (3), we get the following: if u is a k-modified harmonic function so is

$$\begin{aligned}
v(x,y,t) = \tfrac{\partial}{\partial x}\left(\tfrac{1}{r^{k+1}}u\left(\tfrac{x}{r^2}, \tfrac{y}{r^2}, \tfrac{t}{r^2}\right)\right) &= -\tfrac{(k+1)x}{r^{k+3}}u\left(\tfrac{x}{r^2}, \tfrac{y}{r^2}, \tfrac{t}{r^2}\right)\\
&+ \tfrac{1}{r^{k+1}}\left[\tfrac{\partial u}{\partial x}\left(\tfrac{x}{r^2}, \tfrac{y}{r^2}, \tfrac{t}{r^2}\right)\tfrac{-x^2+y^2+t^2}{r^4} - \tfrac{\partial u}{\partial y}\left(\tfrac{x}{r^2}, \tfrac{y}{r^2}, \tfrac{t}{r^2}\right)\tfrac{2xy}{r^4} - \tfrac{\partial u}{\partial t}\left(\tfrac{x}{r^2}, \tfrac{y}{r^2}, \tfrac{t}{r^2}\right)\tfrac{2xt}{r^4}\right].
\end{aligned}$$

Applying again the Kelvin transform, but this time to v, we obtain the following k-modified harmonic function

$$\tfrac{1}{r^{k+1}}v\left(\tfrac{x}{r^2}, \tfrac{y}{r^2}, \tfrac{t}{r^2}\right) = -\left[(k+1)xu(x,y,t) + \tfrac{\partial u}{\partial x}(x,y,t)(x^2 - y^2 - t^2) + \tfrac{\partial u}{\partial y}(x,y,t)2xy + \tfrac{\partial u}{\partial t}(x,y,t)2xt\right].$$

and hence, it is shown that u_3 in the above lemma is indeed k-modified harmonic. A similar argument, this time differentiating with respect to y, yields u_4.

In order to prove Theorem 1.1, it suffices to show that

$$\mathcal{P}_n(\mathbb{R}^3) = \mathcal{H}_n^k(\mathbb{R}^3) \oplus r^2\mathcal{P}_{n-2}(\mathbb{R}^3),\; r^2 = x^2 + y^2 + t^2,\, n \geq 2, \tag{4}$$

since then, with $\mathcal{P}_0(\mathbb{R}^3) = \mathcal{H}_0^k(\mathbb{R}^3)$, and $\mathcal{P}_1(\mathbb{R}^3) = \mathcal{H}_1^k(\mathbb{R}^3)$, we have

$$\mathcal{P}_2(\mathbb{R}^3) = \mathcal{H}_2^k(\mathbb{R}^3) + r^2\mathcal{H}_0^k(\mathbb{R}^3),\ \mathcal{P}_3(\mathbb{R}^3) = \mathcal{H}_3^k(\mathbb{R}^3) + r^2\mathcal{H}_1^k(\mathbb{R}^3),$$
$$\mathcal{P}_4(\mathbb{R}^3) = \mathcal{H}_4^k(\mathbb{R}^3) + r^2(\mathcal{H}_2^k(\mathbb{R}^3) + r^2\mathcal{H}_0^k(\mathbb{R}^3)) = \mathcal{H}_4^k(\mathbb{R}^3) + r^2\mathcal{H}_2^k(\mathbb{R}^3) + r^4\mathcal{H}_0^k(\mathbb{R}^3),$$

and so on.

For the proof of (4), we need the following lemma:

Lemma 1.3. *Assume that the monomial* $x^i y^j t^{2m}$ $(i, j, m \in \mathbb{N}_0)$ *is representable in the form*

$$x^i y^j t^{2m} = u(x, y, t) + r^2 q(x, y, t), \qquad r^2 = x^2 + y^2 + t^2, \tag{5}$$

where $u \in \mathcal{H}_n^k(\mathbb{R}^3)$ *and* $q \in \mathcal{P}_{n-2}(\mathbb{R}^3)$, $n = i + j + 2m$.

Then there are homogeneous k-modified harmonic polynomials v_1, v_2 *of degree* $n+1$ *and polynomials* $q_1, q_2 \in \mathcal{P}_{n-1}(\mathbb{R}^3)$ *such that*

$$x^{i+1} y^j t^{2m} = v_1(x, y, t) + r^2 q_1(x, y, t), \qquad x^i y^{j+1} t^{2m} = v_2(x, y, t) + r^2 q_2(x, y, t).$$

Proof. Applying the operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial t}$ to (5) leads to

$$\frac{\partial u}{\partial x} = ix^{i-1} y^j t^{2m} - 2xq - r^2\frac{\partial q}{\partial x},\ \frac{\partial u}{\partial y} = jx^i y^{j-1} t^{2m} - 2yq - r^2\frac{\partial q}{\partial y},$$
$$\frac{\partial u}{\partial t} = 2mx^i y^j t^{2m-1} - 2tq - r^2\frac{\partial q}{\partial t}.$$

Inserting u from (5) and these three partial derivatives in

$$\begin{aligned} v(x, y, t) :&= (k+1)xu + (x^2 - y^2 - t^2)\frac{\partial u}{\partial x} + 2xy\frac{\partial u}{\partial y} + 2xt\frac{\partial u}{\partial t} \\ &= (k+1)xu + (2x^2 - r^2)\frac{\partial u}{\partial x} + 2xy\frac{\partial u}{\partial y} + 2xt\frac{\partial u}{\partial t}, \end{aligned}$$

one obtains

$$v = (2n+1+k)x^{i+1} y^j t^{2m} - (k+3)xr^2 q - r^2\left[(x^2 - y^2 - t^2)\frac{\partial q}{\partial x} + ix^{i-1} y^j t^{2m} + 2x\left(y\frac{\partial q}{\partial y} + t\frac{\partial q}{\partial t}\right)\right],$$

where $n = i + j + 2m$.

Invoking Euler's relation

$$y\frac{\partial q}{\partial y} + t\frac{\partial q}{\partial t} = (n-2)q - x\frac{\partial q}{\partial x},$$

we find that

$$(2n+1+k)x^{i+1} y^j t^{2m} = v + r^2\left[(2n-1+k)xq + ix^{i-1} y^j t^{2m} - r^2\frac{\partial q}{\partial x}\right].$$

By Lemma 1.2, v and hence also $v_1 = \frac{v}{(2n+1+k)}$ are homogeneous k-modified harmonic polynomials.

Setting

$$q_1 := \frac{1}{(2n+1+k)}\left[(2n-1+k)xq + ix^{i-1} y^j t^{2m} - r^2\frac{\partial q}{\partial x}\right]$$

we see that $q_1 \in \mathcal{P}_{n-1}(\mathbb{R}^3)$, satisfying

$$x^{i+1} y^j t^{2m} = v_1(x, y, t) + r^2 q_1(x, y, t).$$

Similarly we prove the second claim.

By induction on i, applying Lemma 1.3 (with $j = m = 0$), we now find that there are $u \in \mathcal{H}_i^k(\mathbb{R}^3)$ and $q \in \mathcal{P}_{i-2}(\mathbb{R}^3)$ such that

$$x^i = u + r^2 q \ (i \geq 2).$$

Applying Lemma 1.3 again, this time with induction on j (for fixed i), we see that there are $\tilde{u} \in \mathcal{H}^k_{i+j}(\mathbb{R}^3)$ and $\tilde{q} \in \mathcal{P}_{i+j-2}(\mathbb{R}^3)$ such that

$$x^i y^j = \tilde{u} + r^2\tilde{q}. \tag{6}$$

In order to get an analogous result for t^{2m}, we argue as follows: According to (6), there are $v_j \in \mathcal{H}^k_{2m}(\mathbb{R}^3)$ and $q_j \in \mathcal{P}_{2m-2}(\mathbb{R}^3)$ such that

$$x^{2j} y^{2m-2j} = v_j + r^2 q_j. \tag{7}$$

Consequently,

$$\begin{aligned}
t^{2m} &= [r^2 - (x^2+y^2)]^m = \textstyle\sum_{\ell=0}^m (-1)^{m-\ell} \binom{m}{\ell} r^{2\ell}(x^2+y^2)^{m-\ell} \\
&= (-1)^m (x^2+y^2)^m + \textstyle\sum_{\ell=1}^m (-1)^{m-\ell} \binom{m}{\ell} r^{2\ell}(x^2+y^2)^{m-\ell} \\
&= (-1)^m (x^2+y^2)^m + r^2 \textstyle\sum_{\ell=0}^{m-1} (-1)^{m-\ell-1} \binom{m}{\ell+1} r^{2\ell}(x^2+y^2)^{m-\ell-1} \\
&= (-1)^m \textstyle\sum_{\ell=0}^m \binom{m}{\ell} x^{2\ell} y^{2m-2\ell} + r^2 \sum_{\ell=0}^{m-1} \cdots \\
&= (-1)^m \textstyle\sum_{\ell=0}^m \binom{m}{\ell} (v_\ell + r^2 q_\ell) + r^2 \sum_{\ell=0}^{m-1} \cdots \\
&= (-1)^m \textstyle\sum_{\ell=0}^m \binom{m}{\ell} v_\ell \\
&\quad + r^2 \left[(-1)^m \textstyle\sum_{\ell=0}^m \binom{m}{\ell} q_\ell + \sum_{\ell=0}^{m-1} (-1)^{m-\ell-1} \binom{m}{\ell+1} r^{2\ell}(x^2+y^2)^{m-\ell-1} \right].
\end{aligned}$$

Hence, setting

$$v = (-1)^m \sum_{\ell=0}^{m} \binom{m}{\ell} v_\ell,$$

we found $v \in \mathcal{H}^k_{2m}(\mathbb{R}^3)$ and $q \in \mathcal{P}_{2m-2}(\mathbb{R}^3)$ such that

$$t^{2m} = v + r^2 q.$$

By induction on j, taking again into account Lemma 1.3, we see that there are $\tilde{v} \in \mathcal{H}^k_{j+2m}(\mathbb{R}^3)$ and $\tilde{q} \in \mathcal{P}_{j+2m-2}(\mathbb{R}^3)$ such that

$$y^j t^{2m} = \tilde{v} + r^2 \tilde{q}.$$

From this result, applying Lemma 1.3 again, by induction on i, we find $u \in \mathcal{H}^k_{i+j+2m}(\mathbb{R}^3)$ and $q \in \mathcal{P}_{i+j+2m-2}(\mathbb{R}^3)$ such that

$$x^i y^j t^{2m} = u + r^2 q.$$

Consequently (5) holds, finishing the proof of Theorem 1.1. □

2 | THE CASE $k = -1$

We shall now go into more details in case $k = -1$, i.e., we consider the equation

$$t\Delta u - \frac{\partial u}{\partial t} = 0, \tag{8}$$

in $\mathbb{R}^3 = \{(x, y, t)\}$.

Recall that this equation is the Laplace-Beltrami equation associated with the Poincaré metric

$$ds^2 = \frac{dx^2 + dy^2 + dt^2}{t^2}.$$

We are interested in an explicit basis for the vector space $\mathcal{H}^{-1}_n(\mathbb{R}^3)$ of all homogeneous (-1)-modified harmonic polynomials.

Obviously, if u does not depend on t, u is a classical harmonic function of two variables, and hence, there are two linear independent homogeneous polynomial solutions of degree n, namely,

$$H_n^{n-1}(x,y)=\sum_{\ell=0}^{\left[\frac{n}{2}\right]}(-1)^{\ell}\binom{n}{2\ell}x^{n-2\ell}y^{2\ell}\ \text{and}\ H_n^{n}(x,y)=\sum_{\ell=0}^{\left[\frac{n-1}{2}\right]}(-1)^{\ell}\binom{n}{2\ell+1}x^{n-2\ell-1}y^{2\ell+1}. \tag{9}$$

They arose from the expansion of the complex function $(x+iy)^n$.

Further polynomial solutions of (8) can be obtained as follows:

Consider the transformation $v(x,y,t)=\frac{1}{t^2}u(x,y,t)$. Then u solves (8) if and only if v is a solution of the equation

$$t\Delta v+3\frac{\partial v}{\partial t}=0. \tag{10}$$

But in Leutwiler,[9, §4] it has been shown that the homogeneous polynomials

$$h_{n-2}^{i}(x,y,t)=\sum_{p=0}^{\left[\frac{i}{2}\right]}\sum_{q=0}^{\left[\frac{n-2-i}{2}\right]}d_{pq}^{n-2,i}x^{n-2-i-2q}y^{i-2p}t^{2p+2q},\ (n\geq 2) \tag{11}$$

with

$$d_{pq}^{n-2,i}=(-1)^{p+q}\frac{\binom{2p}{p}\binom{2q}{q}\binom{i}{2p}\binom{n-2-i}{2q}}{2^{2(p+q)}(p+q+1)\binom{p+q}{p}},\ 0\leq i\leq n-2,$$

are solutions of (10), more precisely $h_{n-2}^{i}\in\mathcal{H}_n^{3}(\mathbb{R}^3)$, for $0\leq i\leq n-2$. Consequently

$$H_n^{i}(x,y,t)=h_{n-2}^{i}(x,y,t)\cdot t^2,\ 0\leq i\leq n-2,\ (n\geq 2) \tag{12}$$

are homogeneous polynomial solutions of (8). Together with (9), we thus found $n+1$ linear independent polynomials H_n^i, $0\leq i\leq n$, which belong to the vector space $\mathcal{H}_n^{-1}(\mathbb{R}^3)$. In order to show that these polynomials represent a basis of $\mathcal{H}_n^{-1}(\mathbb{R}^3)$, we have to show that

$$\dim\mathcal{H}_n^{-1}(\mathbb{R}^3)=n+1. \tag{13}$$

We argue as follows: Assume that the homogeneous polynomial

$$p(x,y,t)=\sum_{i+j+m=n}a_{ijm}x^iy^jt^m$$

is a solution of (8). Then

$$0=\left(t\Delta p-\frac{\partial p}{\partial t}\right)\bigg|_{t=0}=-\sum_{i+j=n-1}a_{ij1}x^iy^j$$

and thus

$$a_{n-1,0,1}=a_{n-2,1,1}=a_{n-3,2,1}=\ldots=a_{0,n-1,1}=0.$$

There results

$$p(x,y,t)=h(x,y)+t^2q(x,y,t), \tag{14}$$

where

$$h(x,y)=\sum_{i+j=n}a_{i,j,0}x^iy^j$$

and $q(x,y,t)$ is a homogeneous polynomial of degree $n-2$.

The representation (14) inserted into (8) yields the equation

$$\Delta h+t^2\Delta q+3t\frac{\partial q}{\partial t}=0.$$

Evaluated at $t = 0$ we get $\Delta h = 0$, and so $h = h(x,y)$ is homogeneous, classically harmonic polynomial of degree n, i.e., (see (9))

$$h = aH_n^{n-1} + bH_n^n,\ a,b \in \mathbb{R}.$$

Putting $v := p - h = t^2 q$, v solves (8), and hence, q is a homogeneous polynomial of degree $n-2$ which satisfies the equation (10). But in Leutwiler,[9, §4] it has been shown that the homogeneous polynomials h_{n-2}^i, $0 \le i \le n-2, (n \ge 2)$ from (11) form a basis of the vector space $\mathcal{H}_{n-2}^3(\mathbb{R}^3)$ and thus

$$q = \sum_{i=0}^{n-2} a_i h_{n-2}^i \text{ for some } a_i \in \mathbb{R},\ 0 \le i \le n-2. \tag{15}$$

We therefore proved that

$$\begin{aligned} p(x,y,t) &= \sum_{i=0}^{n-2} a_i h_{n-2}^i(x,y,t) \cdot t^2 + a_{n-1} H_n^{n-1}(x,y,t) + a_n H_n^n(x,y,t) \\ &= \sum_{i=0}^{n-2} a_i H_n^i(x,y,t) + a_{n-1} H_n^{n-1}(x,y,t) + a_n H_n^n(x,y,t) = \sum_{i=0}^{n} a_i H_n^i(x,y,t), \end{aligned}$$

showing that the $H_n^i\,(0 \le i \le n)$ form a generating system for the vector space $\mathcal{H}_n^{-1}(\mathbb{R}^3)$, and thus, (13) holds.

We recollect

Theorem 2.1. *The homogeneous (− 1)-modified harmonic polynomials $H_n^i\,(0 \le i \le n)$, defined in (12) and (9), form a basis for the vector space $\mathcal{H}_n^{-1}(\mathbb{R}^3)$.*

3 | THE CASE $k = -2\ell\ (\ell \in \mathbb{N})$

From Leutwiler,[10, §6] we know that the homogeneous polynomials of degree n

$$u_n^i(x,y,t) = \sum_{p=0}^{\left[\frac{i}{2}\right]} \sum_{q=0}^{\left[\frac{n-i}{2}\right]} d_{pq}^{n,i} x^{n-i-2q} y^{i-2p} t^{2p+2q},\ 0 \le i \le n, \tag{16}$$

with $d_{00}^{n,i} = 1$ and

$$d_{pq}^{n,i} = (-1)^{p+q} \frac{p!q! \binom{2p}{p} \binom{2q}{q} \binom{i}{2p} \binom{n-i}{2q}}{2^{p+q}(1-2\ell)(3-2\ell)(5-2\ell)\ \ldots\ [(2(p+q)-1)-2\ell]},\ (p,q) \ne (0,0),$$

are solutions of (1) with $k = -2\ell$, i.e. $u_n^i \in \mathcal{H}_n^{-2\ell}(\mathbb{R}^3)$, for $0 \le i \le n$. But in case $n \ge 2\ell + 1$, these are not all the homogeneous polynomial solutions of $\mathcal{H}_n^{-2\ell}(\mathbb{R}^3)$. Indeed, additional members for this vector space we get from the transformation

$$v(x,y,t) = \frac{1}{t^{2\ell+1}} u(x,y,t).$$

If u is a solution of (1) with $k = -2\ell$, v solves the equation

$$t\Delta v + 2(\ell+1)\frac{\partial v}{\partial t} = 0. \tag{17}$$

Reaching back again Leutwiler,[10, §6] we get for all $n \ge 2\ell + 1$ the following solutions of (17):

$$\tilde{v}_{n-2\ell-1}^i(x,y,t) = \sum_{p=0}^{\left[\frac{i}{2}\right]} \sum_{q=0}^{\left[\frac{n-2\ell-1-i}{2}\right]} d_{pq}^{n-2\ell-1,i} x^{n-2\ell-1-i-2q} y^{i-2p} t^{2p+2q},\ 0 \le i \le n-2\ell-1, \tag{18}$$

with $d_{00}^{n-2\ell-1,i} = 1$, and

$$d_{pq}^{n-2\ell-1,i} = (-1)^{p+q} \frac{p!q! \binom{2p}{p}\binom{2q}{q}\binom{i}{2p}\binom{n-2\ell-1-i}{2q}}{2^{p+q}(3+2\ell)(5+2\ell)(7+2\ell)\ \dots\ [(2(p+q)+1)+2\ell]},$$

$(p,q) \neq (0,0)$. Observing that

$$(1+2\ell)(3+2\ell)(5+2\ell)\ \dots\ [(2m+1)+2\ell] = \frac{[2(m+\ell+1)]!\ell!}{2^{m+1}(m+\ell+1)!(2\ell)!} \tag{19}$$

$d_{pq}^{n-2\ell-1,i}$ can also be written in the form

$$d_{pq}^{n-2\ell-1,i} = (-1)^{p+q} \frac{2p!q! \binom{2p}{p}\binom{2q}{q}\binom{i}{2p}\binom{n-2\ell-1-i}{2q}(p+q+\ell+1)!(2\ell+1)!}{[2(p+q+\ell+1)]!\ell!}.$$

Setting

$$v_n^i(x,y,t) = \tilde{v}_{n-2\ell-1}^i(x,y,t)\cdot t^{2\ell+1},\ 0 \le i \le n-2\ell-1, \tag{20}$$

we thus found another set of homogeneous polynomial solutions of degree n of the equation (1), for $k = -2\ell$, provided $n \geq 2\ell+1$. In other words we proved

Theorem 3.1. *The homogeneous (-2ℓ)-modified harmonic polynomials $u_n^i, 0 \le i \le n$, from (16), together, for $n \geq 2\ell+1$, with the v_n^i, $0 \le i \le n-2\ell-1$, from (20), are members of the vector space $\mathcal{H}_n^{-2\ell}(\mathbb{R}^3)$.*

They probably form a basis for the vector space $\mathcal{H}_n^{-2\ell}(\mathbb{R}^3)$, but we can't show this yet.

4 | THE CASE $k = 0$ OF CLASSICAL HARMONIC FUNCTIONS

Setting $\ell = 0$ in (16), we get the following homogeneous classically harmonic polynomials in $\mathbb{R}^3$:

$$u_n^i(x,y,t) = \sum_{p=0}^{\left[\frac{i}{2}\right]}\sum_{q=0}^{\left[\frac{n-i}{2}\right]} d_{pq}^{n,i} x^{n-i-2q} y^{i-2p} t^{2p+2q},\ 0 \le i \le n, \tag{21}$$

with

$$d_{pq}^{n,i} = (-1)^{p+q} \frac{\binom{2p}{p}\binom{2q}{q}\binom{i}{2p}\binom{n-i}{2q}}{\binom{p+q}{p}\binom{2(p+q)}{p+q}}.$$

From (18), setting $\ell = 0$, we get, with

$$v_n^i(x,y,t) = \tilde{v}_{n-1}^i(x,y,t)\cdot t,\ (n \ge 1), \tag{22}$$

the polynomials

$$v_n^i(x,y,t) = \sum_{p=0}^{\left[\frac{i}{2}\right]}\sum_{q=0}^{\left[\frac{n-1-i}{2}\right]} d_{pq}^{n-1,i} x^{n-1-i-2q} y^{i-2p} t^{2p+2q+1},\ 0 \le i \le n-1, \tag{23}$$

where

$$d_{pq}^{n-1,i} = (-1)^{p+q} \frac{2\binom{2p}{p}\binom{2q}{q}\binom{i}{2p}\binom{n-1-i}{2q}}{(p+q+1)\binom{p+q}{p}\binom{2(p+q+1)}{p+q+1}} = (-1)^{p+q} \frac{\binom{2p}{p}\binom{2q}{q}\binom{i}{2p}\binom{n-1-i}{2q}}{(2p+2q+1)\binom{p+q}{p}\binom{2(p+q)}{p+q}}.$$

The polynomials $u_n^i, 0 \le i \le n$, and $v_n^i, 0 \le i \le n-1$, form a set of $2n+1$ linear independent functions of the vector space $\mathcal{H}_n(\mathbb{R}^3) = \mathcal{H}_n^0(\mathbb{R}^3)$, where $n \geq 1$. Since, e.g., by Axler-Bourdon-Ramey,[12, p82] dim $\mathcal{H}_n(\mathbb{R}^3) = 2n+1$, we have

Theorem 4.1. *The homogeneous harmonic polynomials* u_n^i, $0 \leq i \leq n$, *from (21), together with the homogeneous harmonic polynomials* v_n^i, $0 \leq i \leq n-1$, *from (23), form a basis for the vector space* $\mathcal{H}_n(\mathbb{R}^3)$ *of all homogeneous, classically harmonic polynomials of degree* $n \geq 1$ *in* $\mathbb{R}^3$.

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